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Integral Equations and Operator Theory

Spectra of Toeplitz operators and compositions of Muckenhoupt weights with Blaschke products

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Abstract. We discuss the optimality of a sufficient condition from [12] for a point to belong to the essential spectrum of a Toeplitz operator with a bounded measurable coefficient. Our approach is based on a new sufficient condition for a composition of a Muckenhoupt weight with a Blaschke product to belong to the same Muckenhoupt class.

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1. Introduction and main results

Let $\mathbb{T} = \{\zeta \in \mathbb{C} : \ |\zeta| = 1\}$ be the unit circle. A number $c \in \mathbb{C}$ is called a *(left, right) cluster value* of a measurable function $a : \mathbb{T} \to \mathbb{C}$ at a point $\zeta \in \mathbb{T}$ if $1/(a - c) \notin L^\infty(W)$ for every neighbourhood (left semi-neighbourhood, right semi-neighbourhood) $W \subset \mathbb{T}$ of ζ. Cluster values are invariant under changes of the function on measure zero sets. We denote the set of all left (right) cluster values of a at ζ by $a(\zeta - 0)$ (by $a(\zeta + 0)$), and use also the following notation $a(\zeta) = a(\zeta - 0) \cup a(\zeta + 0)$, $\ a(\mathbb{T}) = \cup_{\zeta\in\mathbb{T}}\, a(\zeta)$. It is easy to see that $a(\zeta - 0)$, $a(\zeta + 0)$, $\ a(\zeta)$ and $a(\mathbb{T})$ are closed sets. Hence they are all compact if $a \in L^\infty(\mathbb{T})$.

Let $H^p(\mathbb{T})$, $1 \le p \le \infty$ denote the Hardy space, that is $H^p(\mathbb{T}) := \{f \in L^p(\mathbb{T}) : \ f_n = 0 \text{ for } n < 0\}$, where f_n is the nth Fourier coefficient of f. Let $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$, $1 < p < \infty$ denote the Toeplitz operator generated by a function $a \in L^\infty(\mathbb{T})$, i.e. $T(a)f = P(af)$, $f \in H^p(\mathbb{T})$, where P is the Riesz

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projection:

$$Pg(\zeta) = \frac{1}{2}\, g(\zeta) + \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{g(w)}{w - \zeta}\, dw, \quad \zeta \in \mathbb{T}.$$

$P : L^p(\mathbb{T}) \to H^p(\mathbb{T})$, $1 < p < \infty$ is a bounded projection and

$$P\left(\sum_{n=-\infty}^{+\infty} g_n \zeta^n\right) = \sum_{n=0}^{+\infty} g_n \zeta^n.$$

If $a(\zeta)$ consists of at most two points for each $\zeta \in \mathbb{T}$, in particular if a is continuous or piecewise continuous, then the spectrum of $T(a)$ can be described in terms of $a(\zeta \pm 0)$, $\zeta \in \mathbb{T}$ (see [3, 4, 13]). This is no longer possible if $a(\zeta)$ is allowed to contain more than two points (see [2, 4.71–4.78] and [10]). It is no longer sufficient to know the values of a in this case, it is important to know "how these values are attained" by a.

Since a complete description of the essential spectrum of $T(a)$ in terms of the cluster values of $a \in L^\infty(\mathbb{T})$ is impossible, it is natural to try finding "optimal" sufficient conditions for a point λ to belong to the essential spectrum. Results of this sort were obtained in [11, 12]. In order to state them we need the following notation.

Let $K \subset \mathbb{C}$ be an arbitrary compact set and $\lambda \in \mathbb{C} \setminus K$. Then the set

$$\sigma(K; \lambda) = \left\{ \frac{w - \lambda}{|w - \lambda|} \;\Big|\; w \in K \right\} \subseteq \mathbb{T}$$

is compact as a continuous image of a compact set. Hence the set $\Delta_\lambda(K) := \mathbb{T} \setminus \sigma(K; \lambda)$ is open in $\mathbb{T}$. So, $\Delta_\lambda(K)$ is the union of an at most countable family of open arcs.

We call an open arc of $\mathbb{T}$ p–large if its length is greater than or equal to $\frac{2\pi}{\max\{p,q\}}$, where $q = \frac{p}{p-1}$, $1 < p < \infty$.

The following result has been proved in [12].

Theorem 1.1. *Let* $1 < p < \infty$, $a \in L^\infty(\mathbb{T})$, $\lambda \in \mathbb{C} \setminus a(\mathbb{T})$ *and suppose that, for some* $\zeta \in \mathbb{T}$,
(i) $\Delta_\lambda(a(\zeta - 0))$ *(or* $\Delta_\lambda(a(\zeta + 0))$*) contains at least two* p*–large arcs,*
(ii) $\Delta_\lambda(a(\zeta + 0))$ *(or* $\Delta_\lambda(a(\zeta - 0))$ *respectively) contains at least one* p*–large arc.*
Then λ *belongs to the essential spectrum of* $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$.

A weaker result (with $\Delta_\lambda(a(\zeta))$ in place of $\Delta_\lambda(a(\zeta \pm 0))$ in condition (ii)) was proved in [11] where it was also shown that condition (i) is optimal in the following sense: for any compact $K \subset \mathbb{C}$ and $\lambda \in \mathbb{C} \setminus K$ such that $\Delta_\lambda(K)$ contains at most one p–large arc there exists $a \in L^\infty(\mathbb{T})$ such that $a(-1 \pm 0) = a(\mathbb{T}) = K$ and $T(a) - \lambda I : H^r(\mathbb{T}) \to H^r(\mathbb{T})$ is invertible for any $r \in [\min\{p,q\}, \max\{p,q\}]$. A question that has been open since [11] is whether or not condition (ii) can be dropped, i.e. whether condition (i) alone is sufficient for λ to belong to the essential spectrum of $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$. The following result gives a negative answer to this question.

Theorem 1.2. *There exists $a \in L^\infty(\mathbb{T})$ such that $a(1-0) = \{\pm 1\}$, $|a| \equiv 1$, $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible for any $p \in (1,2)$, and $T(1/a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible for any $p \in (2,+\infty)$.*

The proof of Theorem 1.2 relies on an argument which is related to the following question. Suppose v is an inner function, i.e. v is a nonconstant function in $H^\infty(\mathbb{T})$ such that $|v| = 1$ almost everywhere on $\mathbb{T}$. If $b \in L^\infty(\mathbb{T})$, then $b \circ v \in L^\infty(\mathbb{T})$ and the question is whether or not the invertibility of $T(b) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ implies that of $T(b \circ v) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$.

An equivalent form of this question is in terms A_p classes (see [1, Section 1]). We say that a measurable function $\rho : \mathbb{T} \to [0, +\infty]$ satisfies the A_p condition if

$$\sup_I \left(\frac{1}{|I|} \int_I \rho^p(\zeta)|d\zeta| \right)^{\frac{1}{p}} \left(\frac{1}{|I|} \int_I \rho^{-q}(\zeta)|d\zeta| \right)^{\frac{1}{q}} = C_p < \infty, \tag{1.1}$$

where $I \subset \mathbb{T}$ is an arbitrary arc and $|I|$ denotes its length. The question is whether or not $\rho \in A_p$ implies $\rho \circ v \in A_p$.

Although the answer is positive in the case $p = 2$ (see, e.g., [1, Section 2]), it turns out that for every $p \in (1, +\infty) \setminus \{2\}$ there exist a Blaschke product B and $\rho \in A_p$ such that $\rho \circ B \notin A_p$ (see [1, Theorem 9]). Equivalently, there exists $b \in L^\infty(\mathbb{T})$ such that $T(b) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible, but $T(b \circ B) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is not invertible (see [1, Theorem 12]).

We prove a result in the opposite direction, namely we describe a class of Blaschke products for which the implications

$$\begin{aligned} &\rho \in A_p \quad \Longrightarrow \quad \rho \circ B \in A_p\,, \\ &T(b) : H^p(\mathbb{T}) \to H^p(\mathbb{T}) \ \text{ is invertible } \ \Longrightarrow \\ &T(b \circ B) : H^p(\mathbb{T}) \to H^p(\mathbb{T}) \ \text{ is invertible} \end{aligned}$$

do hold.

Consider the Blaschke product

$$B\left(e^{i\theta}\right) = \prod_{k=1}^{\infty} \frac{r_k - e^{i\theta}}{1 - r_k e^{i\theta}}\,, \quad \theta \in [-\pi, \pi], \tag{1.2}$$

where $r_k \in (0,1)$ and $\sum_{k=1}^{\infty}(1 - r_k) < 1$.

Theorem 1.3. *Suppose $r_1 \le r_2 \le \cdots \le r_n \le \cdots$, and*

$$\inf_{k \ge 1} \frac{1 - r_{k+1}}{1 - r_k} > 0. \tag{1.3}$$

If ρ satisfies the A_p condition, then $\rho \circ B$ also satisfies the A_p condition.

Corollary 1.4. *Let $1 < p < \infty$, $a \in L^\infty(\mathbb{T})$, and let a Blaschke product B satisfy the conditions of Theorem 1.3. Then $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible if and only if $T(a \circ B) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible.*

Proof. The invertibility of $T(a \circ B)$ implies that of $T(a)$ according to [1, Theorem 12]. The opposite implication follows from Theorem 1.3 (see [1, Section 1]). □

2. Auxiliary results on inner and outer functions

According to the canonical factorisation theorem (see, e.g., [5, Theorem 2.8]), any function from $H^p(\mathbb{T}) \setminus \{0\}$ has a unique, modulo a constant factor, representation as the product of an outer function from $H^p(\mathbb{T})$ and an inner function.

A function $F \in H^p(\mathbb{T})$ is called an *outer function* if

$$F(z) = e^{ic} \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log\phi(t)\, dt\right), \quad |z| < 1, \tag{2.1}$$

where c is a real number, $\phi \geq 0$, $\log\phi \in L^1([-\pi,\pi])$, and $\phi \in L^p([-\pi,\pi])$.

A function $v \in H^\infty(\mathbb{T})$ is called an *inner function* if $|v| = 1$ almost everywhere on $\mathbb{T}$. Any inner function v admits a unique factorisation of the form

$$v(z) = e^{ic} B(z) S(z),$$

where c is a real number, B is a Blaschke product

$$B(z) = z^m \prod_k \frac{\overline{z_k}}{|z_k|} \frac{z_k - z}{1 - \overline{z_k}\, z}$$

with $m \in \mathbb{N}\cup\{0\}$, $z_k = r_k \exp(i\theta_k) \neq 0$, $\theta_k \in (-\pi,\pi]$, $r_k = |z_k| < 1$, $\sum_k (1-r_k) < 1$, and S is a *singular inner function*

$$S(z) = \exp\left(-\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t)\right)$$

with a nonnegative measure μ which is singular with respect to the standard Lebesgue measure on $[-\pi,\pi]$.

We are particularly interested in the case where v has a unique discontinuity at $z = 1$ and infinitely many zeros z_k. In this case, $\lim_{k\to\infty} z_k = 1$, the singular measure μ is supported by the point $t = 0$, and

$$S(z) = \exp\left(\kappa\, \frac{z+1}{z-1}\right), \quad \kappa = \mathrm{const} > 0$$

(see [7, Ch. II, Theorems 6.1 and 6.2]). We will also assume that $B(0) \neq 0$. Then

$$B\left(e^{i\theta}\right) = \prod_{k=1}^{\infty} \frac{\overline{z_k}}{|z_k|} \frac{z_k - e^{i\theta}}{1 - \overline{z_k}\, e^{i\theta}}, \quad \theta \in [-\pi,\pi]. \tag{2.2}$$

Theorem 2.1. *(*[6, Theorem 2.8]*) Suppose B has the form* (2.2) *and* $\lim_{k\to\infty} z_k = 1$. *Then one can choose a branch of* $\arg B\left(e^{i\tau}\right)$ *which is continuous and increasing on* $(0, 2\pi)$*, and which satisfies the following condition*

$$\lim_{\tau\to 0+0} \arg B\left(e^{i\tau}\right) =: A_+ < 0, \qquad \lim_{\tau\to 2\pi-0} \arg B\left(e^{i\tau}\right) =: A_- > 0.$$

Moreover, at lest one of these limits is infinite and

$$\arg B\left(e^{i\theta}\right) = \begin{cases} -2\left(\sum_{\theta_k\geq\theta}(\pi + \varphi_k(\theta)) + \sum_{\theta_k<\theta}\varphi_k(\theta)\right), & \theta \in (0,\pi], \\ \\ 2\left(\sum_{\theta_k\leq\theta}(\pi - \varphi_k(\theta)) - \sum_{\theta_k>\theta}\varphi_k(\theta)\right), & \theta \in [-\pi,0), \end{cases} \tag{2.3}$$

where

$$\varphi_k(\theta) = \arctan\left(\varepsilon_k \cot\frac{\theta-\theta_k}{2}\right), \quad \varepsilon_k = \frac{1-r_k}{1+r_k}\,. \tag{2.4}$$

Theorem 2.2. (See [6, Theorem 2.10 and the end of the proof of Theorem 5.9].) *Suppose a real valued function* η *is continuous on* $[-\pi,\pi]\setminus\{0\}$ *and*

$$\lim_{t\to 0\pm 0}\left(\eta(t)\mp\pi\log|t|\right)=0.$$

Then the function $e^{i\eta}$ *admits the following representation*

$$e^{i\eta(t)} = B\left(e^{it}\right) g\left(B\left(e^{it}\right)\right) d\left(e^{it}\right), \quad t\in[-\pi,\pi],$$

where $g, d\in C(\mathbb{T})$*, the index of* g *is 0, and* B *is the infinite Blaschke product with the zeros*

$$z_k = \frac{2-\exp(-k+1/2)}{2+\exp(-k+1/2)}\,.$$

We finish this section with an example of an outer function which is used in the proof of Theorem 1.2.

Example 2.3. Consider the function

$$h(z) = \exp\left(-i\,c\log\left(i\,\frac{1-z}{2}\right)\right),$$

where $c>0$ and log denotes the branch of logarithm which is analytic in the complex plane cut along $(-\infty,0]$ and real valued on $(0,+\infty)$. It is clear that h is analytic inside the unit disk, and since

$$\operatorname{Im}\left(i\,\frac{1-z}{2}\right) > 0, \quad |z|<1,$$

h satisfies the following estimate

$$1 < |h(z)| < e^{c\pi}, \quad |z|<1.$$

Hence $h, 1/h\in H^\infty(\mathbb{T})$ and h is an outer function (see [7, Ch. II, Corollary 4.7]). It is also clear that $h\in C^\infty(\mathbb{T}\setminus\{1\})$, and since

$$i\,\frac{1-e^{i\theta}}{2} = e^{i\frac{\theta}{2}}\,\sin\frac{\theta}{2}\,,$$

we have

$$|h(e^{i\theta})| = \begin{cases} \exp\left(c\,\frac{\theta}{2}\right), & \theta\in(0,\pi], \\ \exp\left(c\left(\frac{\theta}{2}+\pi\right)\right), & \theta\in[-\pi,0), \end{cases}$$

$$\arg h(e^{i\theta}) = -c\log\left|\sin\frac{\theta}{2}\right|. \tag{2.5}$$

3. Proof of Theorem 1.3

Suppose the conditions of Theorem 1.3 are satisfied and let

$$A(\theta) := \arg B\left(e^{i\theta}\right), \quad A(\pm\pi) = 0.$$

The proof of Theorem 1.3 relies upon analysis of the properties of A. The corresponding results are collected in the following two lemmas. Since A admits the representation (2.3), (2.4) (with $\theta_k = 0$ for all $k = 1, 2, \dots$), it is convenient to rewrite (1.3) in the following equivalent form

$$\inf_{k\ge 1} \frac{\varepsilon_{k+1}}{\varepsilon_k} =: c_0 > 0. \tag{3.1}$$

Lemma 3.1. *a) The derivative A' is increasing on $[-\pi, 0)$ and decreasing on $(0, \pi]$.*
b)[1]

$$\frac{c_1}{4|\sin\frac{\theta}{2}|} \le A'(\theta) \le \frac{|A(\theta)|}{|\sin\theta|}, \quad \forall\theta \in [-\pi,\pi]\setminus\{0\}, \quad c_1 := \min\{c_0, \varepsilon_1\}.$$

c)

$$\frac{A'(\theta/c)}{A'(\theta)} < c^2, \quad \forall\theta\in[-\pi,\pi]\setminus\{0\}, \quad \forall c > 1.$$

Proof. Let

$$A_k(\theta) := \arctan\left(\varepsilon_k \cot\frac{\theta}{2}\right).$$

Then

$$A(\theta) = -2\sum_{k=1}^{\infty} A_k(\theta), \quad A'(\theta) = -2\sum_{k=1}^{\infty} A_k'(\theta)$$

(see (2.3), (2.4)).

a) Since

$$\begin{aligned} -A_k'(\theta) &= \frac{\varepsilon_k}{2\sin^2\frac{\theta}{2}}\, \frac{1}{1+\left(\varepsilon_k\cot\frac{\theta}{2}\right)^2} = \frac{\varepsilon_k}{2\left(\sin^2\frac{\theta}{2} + \left(\varepsilon_k\cos\frac{\theta}{2}\right)^2\right)} \\ &= \frac{\varepsilon_k}{2\left((1-\varepsilon_k^2)\sin^2\frac{\theta}{2} + \varepsilon_k^2\right)}, \end{aligned}$$

A' is increasing on $[-\pi, 0)$ and decreasing on $(0, \pi]$.

b) The equality

$$-A_k'(\theta) = \frac{\varepsilon_k}{2\sin^2\frac{\theta}{2}}\, \frac{1}{1+\left(\varepsilon_k\cot\frac{\theta}{2}\right)^2} = \frac{1}{\sin\theta}\, \frac{\varepsilon_k\cot\frac{\theta}{2}}{1+\left(\varepsilon_k\cot\frac{\theta}{2}\right)^2}$$

implies

$$\left|\frac{A_k'(\theta)}{A_k(\theta)}\right| = \frac{1}{|\sin\theta|}\, \frac{u_k}{(1+u_k^2)\arctan u_k}, \quad u_k = \varepsilon_k\cot\frac{|\theta|}{2}.$$

[1] We will not use the upper estimate for $A'(\theta)$.

Since
$$\sup_{u\in(0,+\infty)} \frac{u}{(1+u^2)\arctan u} = \lim_{u\to 0+0} \frac{u}{(1+u^2)\arctan u} = 1,$$
we get the second inequality in b). Let us prove the first one.

It is clear that
$$A'(\theta) \geq \frac{1}{\sin\theta}\, \frac{\varepsilon_{k_0}\cot\frac{\theta}{2}}{1+\left(\varepsilon_{k_0}\cot\frac{\theta}{2}\right)^2} = \frac{1}{|\sin\theta|}\, \frac{u_{k_0}}{1+u_{k_0}^2}\,, \quad u_{k_0} = \varepsilon_{k_0}\cot\frac{|\theta|}{2}$$
for any $k_0 \in \mathbb{N}$. Let k_0 be the smallest natural number such that $u_{k_0} \leq 1$. If $k_0 > 1$, then (3.1) implies
$$c_0 \leq \frac{\varepsilon_{k_0}}{\varepsilon_{k_0-1}} = \frac{u_{k_0}}{u_{k_0-1}} \leq u_{k_0} \leq 1.$$
Hence
$$\frac{u_{k_0}}{1+u_{k_0}^2} \geq \frac{c_0}{2}$$
and
$$A'(\theta) \geq \frac{c_0}{2|\sin\theta|} \geq \frac{c_0}{4|\sin\frac{\theta}{2}|}\,.$$
If $k_0 = 1$, then
$$A'(\theta) \geq \frac{\varepsilon_1}{2\sin^2\frac{\theta}{2}}\, \frac{1}{1+\left(\varepsilon_1\cot\frac{\theta}{2}\right)^2} \geq \frac{\varepsilon_1}{4\sin^2\frac{\theta}{2}} \geq \frac{\varepsilon_1}{4|\sin\frac{\theta}{2}|}\,.$$
This proves the first inequality in b).

c) Since $\sin\vartheta \leq c\sin\frac{\vartheta}{c}$ and $\cot\frac{\vartheta}{c} > \cot\vartheta$, $\forall\vartheta\in(0,\pi/2]$, we have
$$\frac{A_k'(\theta/c)}{A_k'(\theta)} = \frac{\sin^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2c}}\, \frac{1+\left(\varepsilon_k\cot\frac{\theta}{2}\right)^2}{1+\left(\varepsilon_k\cot\frac{\theta}{2c}\right)^2} < c^2.$$

□

Lemma 3.2. *Suppose* $\vartheta_0,\vartheta_1,\vartheta_2 \in [-\pi,\pi]\setminus\{0\}$, $\operatorname{sign}\vartheta_0 = \operatorname{sign}\vartheta_1 = \operatorname{sign}\vartheta_2$, $|\vartheta_0| > |\vartheta_1| > |\vartheta_2|$, *and*
$$|A(\vartheta_1) - A(\vartheta_0)| = 2\pi = |A(\vartheta_2) - A(\vartheta_1)|.$$
Then
a) $|\vartheta_0 - \vartheta_1| \leq c_2|\vartheta_0|$, *where the constant* $c_2 \in (0,1)$ *depends only on* c_1 *from Lemma* 3.1*-b);*
b)
$$1 \leq \frac{|\vartheta_0-\vartheta_1|}{|\vartheta_1-\vartheta_2|} \leq c_3,$$
where c_3 *depends only on* c_1.

Proof. a) Let $\tilde\vartheta \in (\vartheta_1, \vartheta_0)$ be such that

$$|A(\tilde\vartheta) - A(\vartheta_0)| = \frac{c_1}{4}\,.$$

Then, according to the mean value theorem, there exists $\vartheta^* \in (\tilde\vartheta, \vartheta_0)$ such that

$$\left|A'(\vartheta^*)\left(\tilde\vartheta - \vartheta_0\right)\right| = \frac{c_1}{4}\,.$$

It follows from Lemma 3.1-b) that

$$\frac{c_1}{4\left|\sin\frac{\vartheta^*}{2}\right|}\left|\vartheta_0 - \tilde\vartheta\right| \le \frac{c_1}{4} \quad \Longrightarrow \quad \left|\vartheta_0 - \tilde\vartheta\right| \le \left|\sin\frac{\vartheta^*}{2}\right| \le \left|\sin\frac{\vartheta_0}{2}\right| \le \frac{|\vartheta_0|}{2}\,.$$

Since $\left|\vartheta_0 - \tilde\vartheta\right| \le |\vartheta_0|/2$, the monotonicity of A implies

$$|A(\vartheta_0/2) - A(\vartheta_0)| \ge \frac{c_1}{4}\,.$$

Similarly

$$|A(\vartheta_0/2^j) - A(\vartheta_0/2^{j-1})| \ge \frac{c_1}{4}\,, \quad j \in \mathbb{N}.$$

Let $M = [8\pi/c_1] + 1$. Then

$$|A(\vartheta_0/2^M) - A(\vartheta_0)| = \sum_{j=1}^{M} |A(\vartheta_0/2^j) - A(\vartheta_0/2^{j-1})| \ge M\,\frac{c_1}{4} > \frac{8\pi}{c_1}\,\frac{c_1}{4} = 2\pi.$$

Hence $\vartheta_1 \in (\vartheta_0/2^M, \vartheta_0)$ and

$$|\vartheta_0 - \vartheta_1| < |\vartheta_0 - \vartheta_0/2^M| = \left(1 - 2^{-M}\right)|\vartheta_0|.$$

This proves a) with $c_2 = 1 - 2^{-M} = 1 - 2^{-([8\pi/c_1]+1)}$.

b) According to the mean value theorem, there exist $\varphi_1 \in (\vartheta_1, \vartheta_0)$ and $\varphi_2 \in (\vartheta_2, \vartheta_1)$ such that

$$\frac{|\vartheta_0 - \vartheta_1|}{|\vartheta_1 - \vartheta_2|} = \frac{|A'(\varphi_2)|}{|A'(\varphi_1)|}\,.$$

It follows from part a) that

$$1 \ge \frac{\varphi_2}{\varphi_1} > \frac{\vartheta_2}{\vartheta_0} = \frac{\vartheta_2}{\vartheta_1}\,\frac{\vartheta_1}{\vartheta_0} \ge (1 - c_2)^2 = 2^{-2([8\pi/c_1]+1)}.$$

It is now left to use Lemma 3.1-a), c). One can take $c_3 = 2^{4([8\pi/c_1]+1)}$. □

Proof of Theorem 1.3. Let $\theta_j \in (-\pi, \pi]$ be the points such that

$$A(\theta_j) = -2\pi j, \quad j = 0, \pm1, \pm2, \ldots \tag{3.2}$$

and let

$$I_j = \gamma\left(\exp(i\theta_{j+1}), \exp(i\theta_j)\right), \quad j = 0, \pm1, \pm2, \ldots,$$

where $\gamma(\zeta, \zeta') \subset \mathbb{T}$ is the arc described by a point moving from ζ to ζ' in the counterclockwise direction.

Any arc $I \subset \mathbb{T}$ admits the representation:

$$I = \left(\bigcup_{j \in \mathcal{J}} I_j \right) \bigcup \left(\bigcup_{j \in \tilde{\mathcal{J}}} \tilde{I}_j \right),$$

where the set $\mathcal{J}$ is finite or infinite, the set $\tilde{\mathcal{J}}$ contains at most two elements, and the arcs $\tilde{I}_j$ have one of the following forms:
a) if $\mathcal{J} \neq \emptyset$, then

$$\tilde{I}_j = \gamma \left(\exp(i\theta_j), \exp(i\tilde{\theta}_j) \right) \quad \text{or} \quad \gamma \left(\exp(i\tilde{\theta}_j), \exp(i\theta_j) \right)$$

and

$$|A(\theta_j) - A(\tilde{\theta}_j)| < 2\pi;$$

b) if $\mathcal{J} = \emptyset$, then $\tilde{\mathcal{J}}$ contains one element and

$$\tilde{I}_j = \gamma \left(\exp(i\tilde{\theta}_{j+1}), \exp(i\tilde{\theta}_j) \right),$$

where

$$|A(\tilde{\theta}_{j+1}) - A(\tilde{\theta}_j)| < 4\pi.$$

Case b). Suppose $\mathcal{J} = \emptyset$,

$$I = \tilde{I}_j = \gamma \left(\exp(i\tilde{\theta}_{j+1}), \exp(i\tilde{\theta}_j) \right), \quad |A(\tilde{\theta}_{j+1}) - A(\tilde{\theta}_j)| < 4\pi.$$

Since I may contain the point -1, but does not contain in our case the point 1, it is convenient to switch from the function A defined on $[-\pi, \pi] \setminus \{0\}$ to the following function defined on $(0, 2\pi)$:

$$\mathcal{A}(\psi) = \begin{cases} A(\psi), & \text{if } \psi \in (0, \pi], \\ A(\psi - 2\pi), & \text{if } \psi \in (\pi, 2\pi). \end{cases} \tag{3.3}$$

Let $\psi_0 < \psi_1$ be such that $\mathcal{A}(\psi_0) = A(\tilde{\theta}_{j+1})$ and $\mathcal{A}(\psi_1) = A(\tilde{\theta}_j)$.

Using the change of variable $u = \mathcal{A}(\psi)$ we get

$$\begin{aligned} \Delta_p &:= \frac{1}{|I|} \int_I \rho^p(B(\zeta)) |d\zeta| = \frac{1}{\psi_1 - \psi_0} \int_{\psi_0}^{\psi_1} \rho^p\big(\exp(i\mathcal{A}(\psi))\big) d\psi \\ &= \frac{1}{\psi_1 - \psi_0} \int_{\mathcal{A}(\psi_0)}^{\mathcal{A}(\psi_1)} \rho^p(\exp(iu)) \frac{du}{\mathcal{A}'(\psi(u))} \\ &\le \frac{\max_{\psi \in [\psi_0, \psi_1]} (\mathcal{A}'(\psi))^{-1}}{\psi_1 - \psi_0} \int_{\mathcal{A}(\psi_0)}^{\mathcal{A}(\psi_1)} \rho^p(\exp(iu)) du. \end{aligned}$$

According to the mean value theorem there exists $\psi^* \in (\psi_0, \psi_1)$ such that

$$\mathcal{A}'(\psi^*)(\psi_1 - \psi_0) = \mathcal{A}(\psi_1) - \mathcal{A}(\psi_0).$$

It is now easy to derive from Lemmas 3.1 and 3.2 that

$$\begin{aligned}\Delta_p &\le \frac{\mathcal{A}'(\psi^*)}{\min_{\psi\in[\psi_0,\psi_1]}\mathcal{A}'(\psi)}\left(\frac{1}{\mathcal{A}(\psi_1)-\mathcal{A}(\psi_0)}\int_{\mathcal{A}(\psi_0)}^{\mathcal{A}(\psi_1)}\rho^p(\exp(iu))du\right)\\ &\le \frac{c_4}{\mathcal{A}(\psi_1)-\mathcal{A}(\psi_0)}\int_{\mathcal{A}(\psi_0)}^{\mathcal{A}(\psi_1)}\rho^p(\exp(iu))du,\end{aligned}$$

where the constant c_4 depends only on c_1 from Lemma 3.1-b). Similarly,

$$\frac{1}{|I|}\int_I \rho^{-q}(B(\zeta))|d\zeta| \le \frac{c_4}{\mathcal{A}(\psi_1)-\mathcal{A}(\psi_0)}\int_{\mathcal{A}(\psi_0)}^{\mathcal{A}(\psi_1)}\rho^{-q}(\exp(iu))du.$$

Hence

$$\begin{aligned}&\left(\frac{1}{|I|}\int_I \rho^p(B(\zeta))|d\zeta|\right)^{\frac{1}{p}}\left(\frac{1}{|I|}\int_I \rho^{-q}(B(\zeta))|d\zeta|\right)^{\frac{1}{q}} \le\\ &c_4\left(\frac{1}{\mathcal{A}(\psi_1)-\mathcal{A}(\psi_0)}\int_{\mathcal{A}(\psi_0)}^{\mathcal{A}(\psi_1)}\rho^p(\exp(iu))du\right)^{\frac{1}{p}}\times\\ &\left(\frac{1}{\mathcal{A}(\psi_1)-\mathcal{A}(\psi_0)}\int_{\mathcal{A}(\psi_0)}^{\mathcal{A}(\psi_1)}\rho^{-q}(\exp(iu))du\right)^{\frac{1}{q}} \le 2c_4C_p\end{aligned}$$

(see (1.1)). The factor 2 appears in the right-hand side because $\mathcal{A}(\psi_1)-\mathcal{A}(\psi_0)$ may be larger than 2π but is less than $2\times 2\pi$.

Case a). Let $\mathcal{J}_0 \subset \mathbb{Z}$ be the smallest set such that

$$I \subseteq \bigcup_{j\in\mathcal{J}_0} I_j\,.$$

It follows from Lemma 3.2-b) that

$$\sum_{j\in\mathcal{J}_0}|I_j| \le c_5\sum_{j\in\mathcal{J}}|I_j| \le c_5|I|, \tag{3.4}$$

where the constant c_5 depends only on c_1 from Lemma 3.1-b).

Let us estimate

$$\Lambda_{j,p} = \int_{I_j}\rho^p(B(\zeta))|d\zeta|.$$

This is similar but easier than the estimate for Δ_p in the case b), because we do not need to deal with the function (3.3) now. Since $A(\theta_j)-A(\theta_{j+1}) = 2\pi$, we have

$$\Lambda_{j,p} \le \frac{c_4|I_j|}{2\pi}\int_{-2\pi(j+1)}^{-2\pi j}\rho^p(\exp(iu))du = \frac{c_4|I_j|}{2\pi}\|\rho\|^p_{L^p(\mathbb{T})}.$$

Hence

$$\int_I \rho^p(B(\zeta))|d\zeta| \le \int_{\bigcup_{j\in\mathcal{J}_0} I_j} \rho^p(B(\zeta))|d\zeta| = \sum_{j\in\mathcal{J}_0} \int_{I_j} \rho^p(B(\zeta))|d\zeta|$$

$$\le \sum_{j\in\mathcal{J}_0} \frac{c_4|I_j|}{2\pi} \|\rho\|^p_{L^p(\mathbb{T})} = \frac{c_4}{2\pi} \|\rho\|^p_{L^p(\mathbb{T})} \sum_{j\in\mathcal{J}_0} |I_j| \le \frac{c_4 c_5}{2\pi} \|\rho\|^p_{L^p(\mathbb{T})} |I|$$

(see (3.4)). Similarly

$$\int_I \rho^{-q}(B(\zeta))|d\zeta| \le \frac{c_4 c_5}{2\pi} \|\rho^{-1}\|^q_{L^q(\mathbb{T})} |I|.$$

Hence

$$\left(\frac{1}{|I|}\int_I \rho^p(B(\zeta))|d\zeta|\right)^{\frac{1}{p}} \left(\frac{1}{|I|}\int_I \rho^{-q}(B(\zeta))|d\zeta|\right)^{\frac{1}{q}} \le$$

$$\frac{c_4 c_5}{2\pi} \|\rho\|_{L^p(\mathbb{T})} \|\rho^{-1}\|_{L^q(\mathbb{T})} \le c_4 c_5 C_p\,.$$

□

Remark 3.3. The proof of Theorem 1.3 can be easily extended to any inner function v such that $\arg v(e^{i\tau})$ has a continuous and increasing branch on $(0, 2\pi)$, and $A(\theta) := \arg v(e^{i\theta})$ has the following property

$$\frac{\max_{\theta\in[\theta_{j+1},\theta_{j-1}]} A'(\theta)}{\min_{\theta\in[\theta_{j+1},\theta_{j-1}]} A'(\theta)} \le m < +\infty, \quad \forall j \in \mathbb{Z}, \tag{3.5}$$

where θ_j's are defined by (3.2). Indeed, (3.5) is exactly what is needed for the case b) in the proof of Theorem 1.3. The case a) relies also on Lemma 3.2-b) which in turn follows from (3.5).

The above applies for example to the singular inner function

$$S(\zeta) = \exp\left(\kappa\,\frac{\zeta+1}{\zeta-1}\right), \quad \kappa = \text{const} > 0.$$

Indeed,

$$A(\theta) = \arg S(e^{i\theta}) = -\kappa \cot\frac{\theta}{2}$$

and it is not difficult to see that (3.5) holds in this case. This corresponds to the case of the so-called periodic discontinuity which was considered in [9].

4. Proof of Theorem 1.2

Proof. Let $a_0 \in L^\infty(\mathbb{T})$ be defined by

$$a_0(e^{i\tau}) = \exp\left(i\,\frac{\tau}{2}\right), \quad \tau \in (0, 2\pi).$$

Then a_0 is continuous on $\mathbb{T}\setminus\{1\}$, $a_0(1\pm 0) = \pm 1$, $T(a_0) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible for any $p \in (1, 2)$, and $T(1/a_0) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible for any $p \in (2, +\infty)$ (see [8, 9.3, 9.8] or [2, 5.39]).

Let $h_0 = h \exp\left(-i\frac{\pi}{2}\log 2\right)$, where h is the function from Example 2.3 with $c = \frac{\pi}{2}$. Then

$$h_0(e^{it}) = |h(e^{it})|\, e^{i\varphi(t)}, \quad t \in [-\pi, \pi],$$

where

$$\varphi(t) = -\frac{\pi}{2}\,\log\left|2\sin\frac{t}{2}\right|$$

(see (2.5)).

Let f be a 2π-periodic function such that $f \in C^\infty([-\pi,\pi]\setminus\{0\})$, $f(t) = \varphi(t)$ if $-\pi/2 \le t < 0$, and $f(t) = -f(-t)$ if $0 < t \le \pi/2$. Then

$$e^{2if(t)} = B\left(e^{it}\right) g\left(B\left(e^{it}\right)\right) d\left(e^{it}\right), \quad t \in [-\pi, \pi], \tag{4.1}$$

where $g, d \in C(\mathbb{T})$, the index of g is 0, and B is the infinite Blaschke product with the zeros

$$z_k = \frac{2 - \exp(-k + 1/2)}{2 + \exp(-k + 1/2)}$$

(see Theorem 2.2). Since the index of g is 0, there exists $g_0 \in C(\mathbb{T})$ such that $g_0^2 = g$. Let $d_0 \in C(\mathbb{T})$ be such that $d_0^2(e^{it}) = d(e^{it})$ for $t \in [-\pi/2, \pi/2]$, $d_0(e^{it}) \ne 0$ for $t \in [-\pi, \pi]$ and the index of d_0 is 0.

Consider the function $a \in L^\infty(\mathbb{T})$ defined by

$$a(e^{it}) = a_0\left(B\left(e^{it}\right)\right)\left(\frac{g_0\left(B\left(e^{it}\right)\right) d_0\left(e^{it}\right)|h_0(e^{it})|}{h_0(e^{it})}\right). \tag{4.2}$$

It follows from (4.1) and from the definition of the function f that $a^2(e^{it}) = 1$ if $-\pi/2 \le t < 0$. It is clear that the second factor in the right-hand side of (4.2) is continuous on $\{e^{it} | -\pi/2 \le t < 0\}$, whereas the first one has infinitely many discontinuities in any left semi-neighbourhood of 1. Hence a takes values 1 and -1 in any left semi-neighbourhood of 1. So, $a(1-0) = \{\pm 1\}$.

The operator $T(a^{\pm 1}) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible if and only if $T(a_0^{\pm 1} \circ B) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible (see, e.g., [6, Theorem 2.1, Propositions 2.3, 4.1 and 5.4]). The latter operator is indeed invertible because $T(a_0^{\pm 1}) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible and B satisfies (1.3) (see Corollary 1.4). □

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