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Anatomy of the C^*-algebra generated by Toeplitz operators with piece-wise continuous symbols *

Sergei Grudsky and Nikolai Vasilevski
Departamento de Matemáticas, CINVESTAV
Apartado Postal 14-740, 07000, México, D.F., México
e-mail: grudsky@math.cinvestav.mx, nvasilev@math.cinvestav.mx

Abstract

We study the structure of the C^*-algebra generated by Toeplitz operators with piece-wise continuous symbols, putting a special emphasis to Toeplitz operators with unbounded symbols. We show that none of a finite sum of finite products of the initial generators is a compact perturbation of a Toeplitz operator. At the same time the uniform closure of the set of such sum of products contains a huge amount of Toeplitz operators with bounded and unbounded symbols drastically different from symbols of the initial generators.

1 Preliminaries

In the paper we continue the detailed study of the C^*-algebra generated by Toeplitz operators T_a with piece-wise continuous symbols a acting on the Bergman space $\mathcal{A}^2(\mathbb{D})$ on the unit disk $\mathbb{D}$ in $\mathbb{C}$, which was initiated in [4, 6].

We start by recalling of the necessary definitions and results of [4].

Let $\mathbb{D}$ be the unit disk on the complex plane and $\gamma = \partial\mathbb{D}$ be its boundary. Consider the space $L_2(\mathbb{D})$ with the standard Lebesgue plane measure $dv(z) = dxdy$, $z = x + iy \in \mathbb{D}$, and

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its Bergman subspace $\mathcal{A}^2(\mathbb{D})$ which consists of all functions analytic in $\mathbb{D}$. It is well known that the orthogonal Bergman projection B of $L_2(\mathbb{D})$ onto $\mathcal{A}^2(\mathbb{D})$ has the form

$$(B\varphi)(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{\varphi(\zeta)\, dv(\zeta)}{(1 - z\overline{\zeta})^2}.$$

Given a function $a \in L_\infty$, the Toeplitz operator T_a with symbol a is defined as follows

$$T_a : \; \varphi \in \mathcal{A}^2(\mathbb{D}) \longmapsto B(a\varphi) \in \mathcal{A}^2(\mathbb{D}).$$

As was already mentioned in [6], considering Toeplitz operators with piece-wise continuous symbols, it turns out that both the curves supporting symbol discontinuities and the number of such curves meeting at a boundary point of discontinuity do not play actually any essential role for the Toeplitz operator algebra studied. We can start from very different sets of symbols and obtain exactly the same operator algebra as a result. Thus, without loss of generality, we will use the same setup as in [4].

We fix a finite number of distinct points $T = \{t_1, ..., t_m\}$ on the boundary γ of the unit disk $\mathbb{D}$, and let

$$\delta = \min_{k \neq j}\{|t_k - t_j|,\, 1\}.$$

Denote by ℓ_k, $k = 1, ..., m$, the part of the radius of $\mathbb{D}$ starting at t_k and having length $\delta/3$; and let $\mathcal{L} = \bigcup_{k=1}^{m} \ell_k$. We denote by $PC(\overline{\mathbb{D}}, T)$ the set (algebra) of all functions piece-wise continuous on $\mathbb{D}$ which are continuous in $\overline{\mathbb{D}} \setminus \mathcal{L}$ and have one-sided limit values at every point of $\mathcal{L}$. In particular, every function $a \in PC(\overline{\mathbb{D}}, T)$ has at each point $t_k \in T$ two (different, in general) limit values:

$$a^-(t_k) = a(t_k - 0) = \lim_{\gamma \ni t \to t_k,\, t \prec t_k} a(t) \quad \text{and} \quad a^+(t_k) = a(t_k + 0) = \lim_{\gamma \ni t \to t_k,\, t \succ t_k} a(t),$$

where the signs $\pm$ correspond to the standard orientation of the boundary γ of $\mathbb{D}$.

For each $k = 1, ..., m$, we denote by $\chi_k = \chi_k(z)$ the characteristic function of the half-disk obtained by cutting $\mathbb{D}$ by the diameter passing through $t_k \in T$, and such that $\chi_k^+(t_k) = 1$, and thus $\chi_k^-(t_k) = 0$.

In [4] we define the functions $v_k = v_k(z)$, $k = 1, ..., m$, as follows. For each $k = 1, ..., m$, we introduce two neighborhoods of the point t_k:

$$V_k' = \{z \in \overline{\mathbb{D}} : |z - t_k| < \frac{\delta}{6}\} \quad \text{and} \quad V_k'' = \{z \in \overline{\mathbb{D}} : |z - t_k| < \frac{\delta}{3}\},$$

and fix a continuous function $v_k = v_k(z) : \; \overline{\mathbb{D}} \to [0, 1]$ such that

$$v_k|_{\overline{V_k'}} \equiv 1, \qquad v_k|_{\overline{\mathbb{D}} \setminus V_k''} \equiv 0.$$

But in this paper we need to make the functions $v_k = v_k(z)$ more specific. For each $k = 1, ..., m$, introduce the Möbius transformation

$$\alpha_k(z) = i\frac{t_k - z}{z + t_k}, \tag{1.1}$$

which maps the unit disk $\mathbb{D}$ onto the upper half-plane Π, sending the point t_k to 0 and the opposite point $-t_k$ to ∞. We assume now that each $v_k = v_k(z)$ is a C^∞-function and that the function $v_k(\alpha_k^{-1}(z)) = \widehat{v}_k(r)$ depends only on r, the radial part of a point $z = re^{i\theta} \in \Pi$.

We denote by $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$ the C^*-algebra generated by all Toeplitz operators T_a whose symbols a belong to $PC(\overline{\mathbb{D}}, T)$. It is well known that this algebra is irreducible and contains the entire ideal $\mathcal{K}$ of all compact on $\mathcal{A}^2(\mathbb{D})$ operators.

Recall that the main reason caused a quite complicated structure of $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$ was that the semi-commutator $[T_a, T_b) = T_aT_b - T_{ab}$, for a, $b \in PC(\overline{\mathbb{D}}, T)$, is not compact in general (while the commutator $[T_a, T_b] = T_aT_b - T_bT_a$ is always compact).

This implies that the algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$, apart of its initial generators T_a with $a \in PC(\overline{\mathbb{D}}, T)$, contains all elements of the form

$$\sum_{k=1}^{p} \prod_{j=1}^{q_k} T_{a_{j,k}} \tag{1.2}$$

and the uniform limits of sequences of such elements.

In what follows we will need the description of the (Fredholm) symbol algebra $\operatorname{Sym}\mathcal{T}(PC(\overline{\mathbb{D}}, T)) = \mathcal{T}(PC(\overline{\mathbb{D}}, T))/\mathcal{K}$ of the algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$, which we now proceed to characterize.

Let $\widehat{\gamma}$ be the boundary γ, cut at the points $t_k \in T$. The pair of points of $\widehat{\gamma}$ which correspond to the point $t_k \in T$, $k = 1, ..., m$, will be denoted by $t_k - 0$ and $t_k + 0$, following the positive orientation of γ. Let $\overline{X} = \bigsqcup_{k=1}^{m} \Delta_k$ be the disjoint union of segments $\Delta_k = [0, 1]_k$. Denote by Γ the union $\widehat{\gamma} \cup \overline{X}$ with the following point identification

$$t_k - 0 \equiv 0_k, \qquad t_k + 0 \equiv 1_k,$$

where $t_k \pm 0 \in \widehat{\gamma}$, 0_k and 1_k are the boundary points of Δ_k, $k = 1, ..., m$.

Theorem 1.1 ([7, 8, 9]) *The symbol algebra* $\operatorname{Sym}\mathcal{T}(PC(\overline{\mathbb{D}}, T)) = \mathcal{T}(PC(\overline{\mathbb{D}}, T))/\mathcal{K}$ *of the algebra* $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$ *is isomorphic and isometric to the algebra* $C(\Gamma)$. *The homomorphism*

$$\operatorname{sym} : \mathcal{T}(PC(\overline{\mathbb{D}}, T)) \longrightarrow \operatorname{Sym}\mathcal{T}(PC(\overline{\mathbb{D}}, T)) \cong C(\Gamma)$$

is generated by the mapping of generators of $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$

$$\mathrm{sym}\ : T_a \longmapsto \begin{cases} a(t), & t \in \widehat{\gamma} \\ a(t_k - 0)(1-x) + a(t_k + 0)x, & x \in [0,1]_k \end{cases},$$

where $t_k \in T$, $k = 1, 2, ..., m$.

The following results were, in particular, obtained in [4].

Theorem 1.2 *Each operator $A \in \mathcal{T}(PC(\overline{\mathbb{D}}, T))$ admits the canonical representations*

$$\begin{aligned} A &= T_{s_A} + \sum_{k=1}^{m} T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k} + K \\ &= T_{s_A} + \sum_{k=1}^{m} T_{u_k} f_{A,k}(T_{\chi_k}) + K' \\ &= T_{s_A} + \sum_{k=1}^{m} f_{A,k}(T_{\chi_k}) T_{u_k} + K'', \end{aligned}$$

where $u_k(z) = v_k(z)^2$; K, K', K'' are compact operators,

$$f_{A,k}(x) = (\mathrm{sym}\, A)|_{\Delta_k}, \qquad x \in [0,1]_k, \qquad k = 1, ..., m,$$

$$s_A(t) = (\mathrm{sym}\, A)(t) - \sum_{k=1}^{m} v_k^2(t)[f_{A,k}(0)(1 - \chi_k(t)) + f_{A,k}(1)\chi_k(t)], \tag{1.3}$$

We mention that $s_A(t)$ is a function continuous on γ and that $s_A(t_k) = 0$ for all $t_k \in T$.
Next two theorems characterize Toeplitz operators with bounded measurable symbols in the algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$.

Theorem 1.3 *An operator $A \in \mathcal{T}(PC(\overline{\mathbb{D}}, T))$ is a compact perturbation of a Toeplitz operator if and only if each operator $f_{A,k}(T_{\chi_k})$, $k = 1, ..., m$, is a Toeplitz operator.*

Theorem 1.4 *Let $A = T_a + K \in \mathcal{T}(PC(\overline{\mathbb{D}}, T))$, thus, for each $k = 1, ..., m$, the operator $f_{A,k}(T_{\chi_k})$ is Toeplitz, i.e., $f_{A,k}(T_{\chi_k}) = T_{a_k}$, for some $a_k \in L_\infty(\mathbb{D})$. Then the symbol a of the operator T_a is as follows*

$$a(z) = s_A(z) + \sum_{k=1}^{m} a_k(z) v_k^2(z),$$

where $s_A(z)$ is given by (1.3).

The functions $f_{A,k}(x)$, $x \in [0,1]$, and the symbols $a_k(z)$ of the Toeplitz operators $T_{a_k} = f_{A,k}(T_{\chi_k})$, $k = 1, ..., m$, are connected by the formula

$$f_{A,k}(x) = \frac{2x^2}{\pi} \frac{\ln(1-x) - \ln x}{(1-x) - x} \int_0^\pi \widehat{a}_k(\theta) \left(\frac{1-x}{x}\right)^{\frac{2\theta}{\pi}} d\theta,$$

where $\widehat{a}_k(\theta) = a_k(\alpha_k^{-1}(e^{i\theta}))$, where θ is the angular part of $z = re^{i\theta} \in \Pi$, and α_k is given by (1.1).

Anticipating and motivating a further study we give an example showing how monstrous the symbols of Toeplitz operators from $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$ can be .

Example 1.5 Consider the algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, T_0))$ for the special case of the discontinuity set $T_0 = \{t_1, t_2\}$, where $t_2 = -t_1$. Then the Toeplitz operator

$$T_{\chi_1} = T_s + T_{\chi_1 v_1^2} + T_{(1-\chi_2)v_2^2} + K,$$

where $s(z)$ is a function continuous on $\mathbb{D}$ whose restriction on γ coincides with

$$\chi_1(t) - \chi_1(t)v_1^2(t) - (1-\chi_2(t))v_2^2(t) = \chi_1(t)(1 - v_1^2(t) - v_2^2(t))$$

and K is a compact operator, obviously belongs to the algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, T_0))$. And thus for each function $f(x) \in C[0,1]$ the operator $f(T_{\chi_1})$ belongs to the algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, T_0))$ as well.

Introduce the space $L_2(\Pi)$, with the usual Lebesgue plane measure, and its Bergman subspace $\mathcal{A}^2(\Pi)$ which consists of all functions analytic in Π. For each $t_k \in T$, the operator

$$(V_k\varphi)(z) = -\frac{2it_k}{(z+t_k)^2}\, \varphi\left(\alpha_k(z)\right) \tag{1.4}$$

is obviously the unitary operator both from $L_2(\Pi)$ onto $L_2(\mathbb{D})$, and from $\mathcal{A}^2(\Pi)$ onto $\mathcal{A}^2(\mathbb{D})$, and its inverse (and adjoint) has the form

$$(V_k^{-1}\varphi)(w) = -\frac{2it_k}{(w+i)^2}\, \varphi\left(\alpha_k^{-1}(w)\right).$$

It is obvious that

$$V_k T_{\chi_k} V_k^{-1} = T_{\chi_+},$$

where χ_+ is the characteristic function of the right quarter-plane in Π, and that this unitary equivalence implies that

$$f(T_{\chi_k}) = V_k^{-1} f(T_{\chi_+}) V_k \tag{1.5}$$

Now for $t_1 \in T_0$, let $a_0(z)$ be a function on the unit disk such that

$$\widehat{a}_0(\theta) = a_0(\alpha_1^{-1}(e^{i\theta})) = (\sin\theta)^{-\beta} \sin(\sin\theta)^{-\alpha},$$

where $0 \leq \beta < 1$ and $\alpha > 0$.

By Example 6.4 of [4] the Toeplitz operator $T_{\widehat{a}_0}$ is bounded on $\mathcal{A}^2(\Pi)$ and belongs to the algebra generated by T_{χ_+} Moreover for the function

$$f_0(x) = \frac{2x^2}{\pi} \frac{\ln(1-x) - \ln x}{(1-x) - x} \int_0^\pi (\sin\theta)^{-\beta} \sin(\sin\theta)^{-\alpha} \left(\frac{1-x}{x}\right)^{\frac{2\theta}{\pi}} d\theta,$$

which belongs to $C[0,1]$ and obeys the property $f_0(0) = f_0(1) = 0$, we have that $T_{\widehat{a}_0} = f_0(T_{\chi_+})$. Thus the Toeplitz operator

$$T_{a_0} = f_0(T_{\chi_1}) = V_1^{-1} f_0(T_{\chi_+}) V_1 = V_1^{-1} T_{\widehat{a}_0} V_1$$

belongs to the algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, T_0))$.

We note that the symbol $a_0(z)$ is quite horrible, being *unbounded and oscillating* near every point of $\gamma \setminus T$ and having quite a complicated angular behavior approaching the points of T. At the same time the (Fredholm) symbol of the operator T_{a_0} has quite a respectable form:

$$\operatorname{sym} T_{a_0} = \begin{cases} 0, & t \in \widehat{\gamma} \\ f_0(x), & x \in \Delta_1 = [0,1] \\ f_0(1-x), & x \in \Delta_2 = [0,1] \end{cases} .$$

We describe now some results of [10] which we will use in the paper.
Passing to polar coordinates on the upper half-plane Π we have

$$L_2(\Pi) = L_2(\mathbb{R}_+, rdr) \otimes L_2([0,\pi], d\theta) := L_2(\mathbb{R}_+, rdr) \otimes L_2(0,\pi).$$

We introduce two operators: the unitary operator

$$U = M \otimes I : L_2(\mathbb{R}_+, rdr) \otimes L_2(0,\pi) \longrightarrow L_2(\mathbb{R}) \otimes L_2(0,\pi),$$

where the Mellin transform $M : L_2(\mathbb{R}_+, rdr) \longrightarrow L_2(\mathbb{R})$ is given by

$$(M\psi)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} r^{-i\lambda}\, \psi(r)\, dr,$$

and the isometric imbedding $R_0 : L_2(\mathbb{R}) \longrightarrow \mathcal{A}_1^2 \subset L_2(\mathbb{R} \times [0,\pi])$, which is given by

$$(R_0 f)(\lambda,\theta) = f(\lambda) \cdot \sqrt{\frac{2\lambda}{1-e^{-2\pi\lambda}}}\; e^{-(\lambda+i)\theta}.$$

The adjoint operator $R_0^* : L_2(\mathbb{R} \times [0,\pi]) \longrightarrow L_2(\mathbb{R})$ has the form

$$(R_0^*\psi)(\lambda) = \sqrt{\frac{2\lambda}{1-e^{-2\pi\lambda}}} \int_0^\pi \psi(\lambda,\theta)\, e^{-(\lambda-i)\theta}\, d\theta.$$

Now the operator $R = R_0^* U$ maps the space $L_2(\Pi)$ onto $L_2(\mathbb{R})$, and its restriction

$$R|_{\mathcal{A}^2(\Pi)} : \mathcal{A}^2(\Pi) \longrightarrow L_2(\mathbb{R})$$

is an isometric isomorphism. The adjoint operator

$$R^* = U^* R_0 : L_2(\mathbb{R}) \longrightarrow \mathcal{A}^2(\Pi) \subset L_2(\Pi)$$

is an isometric isomorphism of $L_2(\mathbb{R})$ onto the Bergman subspace $\mathcal{A}^2(\Pi)$ of the space $L_2(\Pi)$. We have

$$RR^* = I : \; L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R}) \qquad \text{and} \qquad R^*R = B_\Pi : \; L_2(\Pi) \longrightarrow \mathcal{A}^2(\Pi),$$

where B_Π is the orthogonal Bergman projection of $L_2(\Pi)$ onto $\mathcal{A}^2(\Pi)$.

Denote by $H(L_1(0,\pi))$ the space of all functions homogeneous of zero order on the upper half-plane whose restrictions onto the upper half of the unit circle (angle parameterized by $\theta \in (0,\pi)$) belong to $L_1(0,\pi)$. Writing $a = a(\theta)$ we will often mean both a function from $L_1(0,\pi)$ and its homogeneous extention on the upper half-plane.

Theorem 1.6 ([10]) *Let $a = a(\theta) \in H(L_1(0,\pi))$ such that the Toeplitz operator T_a is bounded. Then T_a, acting on $\mathcal{A}^2(\Pi)$, is unitary equivalent to the multiplication operator $\gamma_a I = R T_a R^*$, acting on $L_2(\mathbb{R})$. The function $\gamma_a(\lambda)$ is given by*

$$\gamma_a(\lambda) = \frac{2\lambda}{1-e^{-2\pi\lambda}} \int_0^\pi a(\theta)\, e^{-2\lambda\theta}\, d\theta, \qquad \lambda \in \mathbb{R}. \tag{1.6}$$

In particular, for $a = \chi_+(\theta)$, we have (see [6])

$$\gamma_{\chi_+}(\lambda) = \frac{1}{e^{-\pi\lambda}+1}, \qquad \lambda \in \mathbb{R}, \tag{1.7}$$

and

$$T_{\chi_+} = R^* \gamma_{\chi_+}(\lambda) R.$$

We mention as well, see for details [6], that the C^*-algebra with identity $\mathcal{T}_+$ generated by the Toeplitz operator T_{χ_+} is isomorphic and isomorphic to $C(\overline{\mathbb{R}})$, and that this isomorphism is generated by the assignment

$$T_{\chi_+} \longmapsto \gamma_{\chi_+}(\lambda).$$

In particular this implies that for every Toeplitz operator T_a, with $a = a(\theta) \in H(L_1(0, \pi))$ in the algebra $\mathcal{T}_+$ the corresponding function $\gamma_a(\lambda)$, given by (1.6), must belong to $C(\overline{\mathbb{R}})$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is the two point compactification of $\mathbb{R}$.

We note that for general symbols $c = c(r, \theta)$ the Toeplitz operator T_c is no longer unitary equivalent to a multiplication operator. The operator RT_cR^* now has a much more complicated structure: it turns out to be a pseudodifferential operator with a certain compount (or double) symbol. The next theorem clarifies this statement for bounded symbols of a special and important case: $c = c(r, \theta) = a(\theta)v(r)$. The case of unbounded $a(\theta)$ will be treated in Theorem 2.4.

Theorem 1.7 *Given a bounded symbol $a(\theta)v(r)$, the Toeplitz operator T_{av} acting on $\mathcal{A}^2(\Pi)$ is unitary equivalent to the pseudodifferential operator $A_1 = RT_{av}R^*$, acting on $L_2(\mathbb{R})$. The operator A_1 is given by*

$$(A_1 f)(\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}} d\xi \int_{\mathbb{R}} a_1(x, y, \xi) e^{i(x-y)\xi} f(y) dy, \qquad x \in \mathbb{R}, \tag{1.8}$$

where its compound symbol $a_1(x, y, \xi)$ has the form

$$a_1(x, y, \xi) = c(x, y)\, \gamma_a\left(\frac{x+y}{2}\right) \widetilde{v}(\xi)$$

with

$$c(x, y) = \frac{1 - e^{-\pi(x+y)}}{x+y} \sqrt{\frac{2x}{1 - e^{-2\pi x}}} \sqrt{\frac{2y}{1 - e^{-2\pi y}}}, \tag{1.9}$$

and $\widetilde{v}(\xi) = v(e^{-\xi})$.

Proof. We have

$$\begin{aligned}
(A_1 f)(\lambda) &= (RT_{a(\theta)v(r)}R^* f)(\lambda) = (R(R^*R)a(\theta)v(r)(R^*R)R^* f)(\lambda) \\
&= ((RR^*)Ra(\theta)v(r)R^*(RR^*)f)(\lambda) = (Ra(\theta)v(r)R^* f)(\lambda) \\
&= (R_0^* a(\theta)(M \otimes I)v(r)(M^{-1} \otimes I)R_0 f)(\lambda)
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\frac{2\lambda}{1-e^{-2\pi\lambda}}} \int_0^\pi e^{-(\lambda-i)\theta} a(\theta)d\theta \, \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} r^{-i\lambda} v(r)dr \\
&\quad \cdot \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} r^{i\alpha-1} f(\alpha) \sqrt{\frac{2\alpha}{1-e^{-2\pi\alpha}}} \, e^{-(\alpha+i)\theta} d\alpha \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} r^{-i\lambda} v(r)dr \, \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} r^{i\alpha-1} f(\alpha)d\alpha \\
&\quad \cdot \sqrt{\frac{2\lambda}{1-e^{-2\pi\lambda}}} \sqrt{\frac{2\alpha}{1-e^{-2\pi\alpha}}} \int_0^\pi e^{-(\lambda+\alpha)\theta} \, a(\theta)d\theta.
\end{aligned}
$$

The last integral gives

$$
\int_0^\pi e^{-(\lambda+\alpha)\theta} \, a(\theta)d\theta = \frac{1-e^{-\pi(\lambda+\alpha)}}{\lambda+\alpha} \, \gamma_a \left(\frac{\lambda+\alpha}{2}\right),
$$

and thus we have

$$
\begin{aligned}
(A_1 f)(\lambda) &= (RT_{a(\theta)v(r)} R^* f)(\lambda) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}_+} dr \int_{\mathbb{R}} c(\lambda, \alpha) \, \gamma_a \left(\frac{\lambda+\alpha}{2}\right) \, v(r) \, r^{-i(\lambda-\alpha)-1} f(\alpha)d\alpha,
\end{aligned}
$$

where

$$
c(\lambda, \alpha) = \frac{1-e^{-\pi(\lambda+\alpha)}}{\lambda+\alpha} \sqrt{\frac{2\lambda}{1-e^{-2\pi\lambda}}} \sqrt{\frac{2\alpha}{1-e^{-2\pi\alpha}}}. \tag{1.10}
$$

Changing variables, $\lambda = x$, $\alpha = y$, and $r = e^{-\xi}$, we finally have

$$
(A_1 f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} d\xi \int_{\mathbb{R}} a_1(x, y, \xi) e^{i(x-y)\xi} f(y)dy, \qquad x \in \mathbb{R},
$$

with

$$
a_1(x, y, \xi) = c(x, y) \, \gamma_a \left(\frac{x+y}{2}\right) \, \widetilde{v}(\xi),
$$

where $c(x, y)$ is given by (1.10), and $\widetilde{v}(\xi) = v(e^{-\xi})$. □

2 Semi-commutators involving unbounded symbols

The following classical semi-commutator property

$$
[T_a, T_b) = T_a T_b - T_{ab} \in \mathcal{K}, \qquad \text{for all} \quad a \in L_\infty(\mathbb{D}), \ b \in C(\overline{\mathbb{D}}),
$$

played an essential role in [4]. The question on compactness of the semicommutator for *unbounded* a is quite delicate and does not have any universal answer. At the next two examples we show that the compactness result does not valid for general, and even special, unbounded symbols a and arbitrary $b \in C(\overline{\mathbb{D}})$. At the same time we will prove it for a certain special and important for us case of unbounded symbols a and a special chose of $b \in C(\overline{\mathbb{D}})$.

The first example is a minor modification of Example 7 from [3], to which we address for further details.

Example 2.1 Let

$$a(z) = a(r) = (1-r^2)^{-\beta} \sin(1-r^2)^{-\alpha} \in L_1(\mathbb{D})$$

and

$$b(z) = b(r) = (1-r^2)^{\varepsilon} \sin(1-r^2)^{-\alpha} \in C(\overline{\mathbb{D}})$$

where $z = re^{i\theta}$, $0 < \varepsilon < \beta < 1$. Then both T_a and T_b are bounded and compact.

The product ab has the form

$$a(r)b(r) = \frac{(1-r^2)^{-(\beta-\varepsilon)}}{2} - \frac{(1-r^2)^{-(\beta-\varepsilon)} \cos 2(1-r^2)^{-\alpha}}{2} = c_1(r) - c_2(r).$$

Then the operator T_{c_1} is unbounded, while the operator T_{c_2} is compact. That is, the operator T_{ab} is not bounded, and the (unbounded) semi-commutator is not compact.

In what follows we will deal with the class of unbounded symbols which, considered in the upper half-plane setting, are the functions $a(\theta) \in H(L_1(0,\pi))$, where $z = re^{i\theta} \in \Pi$, for which the corresponding Toeplitz operators T_a are bounded.

The second example shows that even for such specific symbols $a(\theta)$ the semi-commutator is not compact for each $b(z) \in C(\overline{\Pi})$.

Example 2.2 Let

$$a(z) = a(\theta) = \theta^{-\beta} \sin \theta^{-\alpha}$$

and

$$b(z) = w(r)\, \theta^{\varepsilon} \sin \theta^{-\alpha},$$

where $z = re^{i\theta}$, $0 < \varepsilon < \beta < 1$, $\alpha > 0$, and $w(r)$ is a $[0,1]$-valued C^{∞}- function such that

$$w(r) \equiv \begin{cases} 0, & r \in [0, \delta_1] \\ 1, & r \in [\delta_2, \delta_3] \\ 0, & r \in [\delta_4, +\infty] \end{cases},$$

and $0 < \delta_1 < \delta_2 < \delta_3 < \delta_4 < +\infty$.

The operator T_a is bounded by results of Example 6.3 of [4]; the operator T_b is bounded as well because of $b(z) \in C(\overline{\Pi})$. The product ab has the form

$$a(\theta)b(z) = \frac{w(r)\theta^{-\delta}}{2} - \frac{w(r)\theta^{-\delta}\cos 2\theta^{-\alpha}}{2} = c_1(z) - c_2(z),$$

where $\delta = \beta - \varepsilon \in (0,1)$.

The Toeplitz operator T_{c_2} is bounded by Theorem 2.4. To prove that the semi-commutator $[T_a, T_b)$ is not compact, it is sufficient to show, for example, that the operator T_{c_1} is unbounded. Let $a_\delta(\theta) = \theta^{-\delta}$, then

$$\gamma_{a_\delta}(\lambda) = \frac{2\lambda}{1-e^{-2\pi\lambda}} \int_0^\pi \theta^{-\delta} e^{-2\lambda\theta} d\theta = \frac{(2\lambda)^\delta}{1-e^{-2\pi\lambda}} \int_0^{2\pi\lambda} u^{-\delta} e^{-u} du.$$

It is clear that if $\lambda \to +\infty$ than we have the asymptotics

$$\gamma_{a_\delta}(\lambda) = c_0\lambda^\delta + o(1), \tag{2.1}$$

$$\frac{\partial\gamma_{a_\delta}(\lambda)}{\partial\lambda} = \delta c_0 \lambda^{\delta-1} + o(1), \tag{2.2}$$

where $c_0 = 2^\delta \Gamma(1-\delta)$.

We will use now the representation (1.8) for the operator $A_1 = RT_{c_1}R^*$. Denoting

$$\widehat{w}(x-y) = \frac{1}{2\pi} \int_{\mathbb{R}} \widetilde{w}(\xi) e^{i(x-y)\xi}\, d\xi,$$

where $\widetilde{w}(\xi) = w(e^{-\xi})$, we have

$$(A_1 f)(x) = \int_{\mathbb{R}} c(x,y)\gamma_{a_\delta}\left(\frac{x+y}{2}\right) \widehat{w}(x-y) f(y) dy,$$

where the function $c(x,y)$ is given by (1.9).

We show now that the operator A_1 is unbounded on $L_2(\mathbb{R})$. Introduce the family of functions

$$f_{x_0}(y) = \begin{cases} \varepsilon^{-1/2}, & y \in I_\varepsilon = [x_0 - \varepsilon/2, x_0 + \varepsilon/2] \\ 0, & y \in \mathbb{R} \setminus I_\varepsilon \end{cases},$$

where $\varepsilon = \varepsilon(x_0) = x_0^{-\delta/2}$. It is clear that $\|f_{x_0}\|_{L_2(\mathbb{R})} = 1$.

Let $x \in I_\varepsilon$; denoting

$$K(x,y) = c(x,y)\gamma_{a_\delta}\left(\frac{x+y}{2}\right) \widehat{w}(x-y)$$

we have

$$\begin{aligned}(A_1 f_{x_0})(x) &= \varepsilon^{-1/2}\int_{x_0-\varepsilon/2}^{x_0+\varepsilon/2} K(x,y)dy \\ &= \varepsilon^{1/2}K(x,x)+\varepsilon^{-1/2}\int_{x_0-\varepsilon/2}^{x_0+\varepsilon/2}(K(x,y)-K(x,x))dy = I_1(x)+I_2(x).\end{aligned}$$

When $x_0 \to +\infty$, for the first summand we have

$$\begin{aligned}I_1(x) &= 1\cdot\gamma_{a_\delta}(x)\cdot\widehat{w}(0)\cdot\varepsilon^{1/2}(x_0) \\ &= \widehat{w}(0)c_0\left(x^\delta\cdot x_0^{-\delta/4}+o(1)\right)=\widehat{w}(0)c_0\left(x_0^{3\delta/4}+o(1)\right). \end{aligned} \tag{2.3}$$

As $\widetilde{w}(\xi)\geq 0$, we have that $\widetilde{w}(0)>0$.

Now for the second summand we have

$$|I_2(x)|\leq \varepsilon^{3/2}\sup_{y\in I_\varepsilon}\left|\frac{\partial K}{\partial y}(x,y)\right|.$$

Both functions $\frac{\partial c}{\partial y}(x,y)$ and $\frac{\partial \widehat{w}}{\partial y}(x-y)$ are uniformly bounded on x. The former is bounded by Theorem 4.2, while the latter is bounded as the Fourier transform of a function with a compact support. Thus we have that

$$|I_2(x)|\leq \mathrm{const}\,\varepsilon^{3/2}\sup_{y\in I_\varepsilon}\left(\left|\frac{\partial\gamma_{a_\delta}}{\partial y}\left(\frac{x+y}{2}\right)\right|+\left|\gamma_{a_\delta}\left(\frac{x+y}{2}\right)\right|\right).$$

Asymptotics (2.1) and (2.2) imply that for $x_0\to+\infty$ we have

$$|I_2(x)|\leq \mathrm{const}\,\varepsilon^{3/2}x^\delta \leq \mathrm{const}\,\left(x_0^{-\delta/2}\right)^{3/2}x_0^\delta = \mathrm{const}\ x_0^{\delta/4}. \tag{2.4}$$

Comparing (2.3) and (2.4), for sufficiently large x_0 and $x\in I_\varepsilon$, we have that

$$|(A_1 f_{x_0})(x)|\geq \frac{|\widehat{w}(0)|\,c_0}{2}\,x_0^{3\delta/4}.$$

Thus

$$\begin{aligned}\|A_1 f_{x_0}\|_{L_2(\mathbb{R})} &\geq \left(\left(\frac{|\widehat{w}(0)|\,c_0}{2}\,x_0^{3\delta/4}\right)^2\int_{x_0-\varepsilon/2}^{x_0+\varepsilon/2}dx\right)^{1/2} \\ &\geq \mathrm{const}\,\left(x_0^{3\delta/2}\cdot\varepsilon(x_0)\right)^{1/2} = \mathrm{const}\ x_0^{\delta/2}.\end{aligned}$$

This obviously yields unboundedness of the operator A_1, which in turn implies unboundedness of $T_a b$.

Now as a special choice of functions continuous on $\overline{\mathbb{D}}$ we select any $v_k(z)$, $k = 1, 2, ..., m$, considered in the upper half-plane setting as a function $v = v(r)$, where $z = re^{i\theta} \in \Pi$, as introduced in Section 1. That is, v is a $[0,1]$-valued C^∞-function such that for some $0 < \delta_1 < \delta_2 < +\infty$, we have

$$v(r) \equiv \begin{cases} 1, & r \in [0, \delta_1] \\ 0, & r \in [\delta_2, +\infty] \end{cases}. \tag{2.5}$$

Our aim is to prove that for each $a(\theta) \in H(L_1(0,\pi))$, for which the corresponding Toeplitz operator T_a is bounded, the semi-commutator $T_aT_v - T_{av}$ is compact. To do this we first represent the operators T_aT_v and T_{av} in the form of pseudodifferential operators with certain compound (or double) symbols and then use the next result, which can be found, for example, in [5, Theorem 4.2 and Theorem 4.4].

Denote by $V(\mathbb{R})$ the set of all absolutely continuous functions on $\mathbb{R}$ of bounded total variation, and by $C_b(\mathbb{R}^2, V(\mathbb{R}))$ the set of all functions $a: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}$ such that $u \mapsto a(u,\cdot)$ is a bounded continuous $V(\mathbb{R})$-valued function on $\mathbb{R}^2$. Then, for $a \in C_b(\mathbb{R}^2, V(\mathbb{R}))$, we define

$$cm_u^C(a) = \max\left\{\|a(u+\Delta u,\cdot) - a(u,\cdot)\|_C : \ \Delta u \in \mathbb{R}^2, \ \|\Delta u\| \le 1\right\},$$

and denote by $\mathcal{E}_2^C$ the subset of all functions in $C_b(\mathbb{R}^2, V(\mathbb{R}))$ such that the $V(\mathbb{R})$-valued function $u \mapsto a(u,\cdot)$ is uniformly continuous on $\mathbb{R}^2$ and the following conditions hold,

$$\lim_{\|u\|\to\infty} cm_u^C(a) = 0 \quad \text{and} \quad \lim_{|h|\to 0} \sup_{u\in\mathbb{R}^2} \|a(u,\cdot) - a^h(u,\cdot)\|_V = 0, \tag{2.6}$$

where $a^h(u,\cdot) = a(u, \xi + h)$, for all $(u,\xi) \in \mathbb{R}^2 \times \mathbb{R}$.

Theorem 2.3 ([5]) *If $\partial_\xi^j \partial_y^k a(x,y,\xi) \in C_b(\mathbb{R}\times\mathbb{R}, V(\mathbb{R}))$ for all $k, j = 0, 1, 2$, then the pseudodifferential operator A with compound symbol $a(x,y,\xi)$ defined on functions $f \in C_0^\infty(\mathbb{R})$ by the iterated integral*

$$(Af)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} d\xi \int_{\mathbb{R}} a(x,y,\xi) e^{i(x-y)\xi} f(y)dy, \qquad x \in \mathbb{R}, \tag{2.7}$$

extends to a bounded linear operator on every Lebesgue space $L_p(\mathbb{R})$, $p \in (1,\infty)$.

If $\partial_\xi^j \partial_y^k a(x,y,\xi) \in \mathcal{E}_2^C$ for all $k, j = 0, 1, 2$, then the pseudodifferential operator (2.7) with compound symbol

$$r(x,y,\xi) = a(x,y,\xi) - a(x,x,\xi)$$

is compact on every Lebesgue space $L_p(\mathbb{R})$, $p \in (1,\infty)$.

Considering semicommutators, we prove first that for our selection of symbols $a(\theta)$ and $v(r)$ the Toeplitz operator T_{av} is bounded.

Theorem 2.4 *For each $a(\theta) \in H(L_1(0,\pi))$ such that the Toeplitz operator T_a is bounded and the $[0,1]$-valued C^∞-function $v = v(r)$ of the form (2.5), the Toeplitz operator T_{av} is bounded on $\mathcal{A}^2(\Pi)$.*

Proof. We mention first that the boundedness of T_a is equivalent (by Theorem 1.6) to the boundedness of the corresponding function

$$\gamma_a(\lambda) = \frac{2\lambda}{1-e^{-2\pi\lambda}} \int_0^\pi a(\theta)\, e^{-2\lambda\theta}\, d\theta, \qquad \lambda \in \mathbb{R}.$$

The C^∞-functions with compact support in $\mathbb{R}_+$ obviously form a dense set in $L_2(\mathbb{R}_+)$. Taking any such function f we consider

$$(A_1 f)(\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}} d\xi \int_{\mathbb{R}} a_1(x,y,\xi) e^{i(x-y)\xi} f(y) dy, \qquad x \in \mathbb{R},$$

where the compound symbol $a_1(x,y,\xi)$ has the form

$$a_1(x,y,\xi) = c(x,y)\, \gamma_a\left(\frac{x+y}{2}\right) \widetilde{v}(\xi)$$

with

$$c(x,y) = \frac{1-e^{-\pi(x+y)}}{x+y} \sqrt{\frac{2x}{1-e^{-2\pi x}}} \sqrt{\frac{2y}{1-e^{-2\pi y}}},$$

and $\widetilde{v}(\xi) = v(e^{-\xi})$. We note that $c(x,x) \equiv 1$.

The boundedness of the operator A_1 follows from Theorem 2.3, Theorems 4.1 - 4.4, and the fact that $\widetilde{v}(\xi)$ is a C^∞-function with a compact support.

By the calculations of Theorem 1.7 we have that $T_{av} = R^* A_1 R$. Thus the Toeplitz operator T_{av} is bounded on $\mathcal{A}^2(\Pi)$. □

Now we are ready to prove that the semicommutator $T_a T_v - T_{av}$ is compact.

Theorem 2.5 *For each $a(\theta) \in H(L_1(0,\pi))$ such that the Toeplitz operator T_a is bounded and the $[0,1]$-valued C^∞-function $v = v(r)$ of the form (2.5), the semicommutator $T_a T_v - T_{av}$ is compact.*

PROOF. Calculation analogous to that of Theorem 1.7 yield

$$\begin{aligned}
(A_2 f)(x) &= RT_aT_vR^*f = (RaR^*)(RvR^*)f \\
&= \gamma_a(x)\,(RvR^*)f = \frac{1}{2\pi}\int_{\mathbb{R}} d\xi \int_{\mathbb{R}} a_2(x,y,\xi)e^{i(x-y)\xi}f(y)dy, \qquad x \in \mathbb{R},
\end{aligned}$$

with

$$a_2(x,y,\xi) = c(x,y)\,\gamma_a(x)\,\widetilde{v}(\xi),$$

where $c(x,y)$ is given by (1.9), and $\widetilde{v}(\xi) = v(e^{-\xi})$.

Thus the operator $R^*(T_{av} - T_aT_v)R = A_1 - A_2$ can be represented as a difference of two pseudodifferential operators having the compound symbols

$$\begin{aligned}
r_1(x,y,\xi) &= a_1(x,y,\xi) - a_1(x,x,\xi) \\
&= c(x,y)\,\gamma_a\left(\frac{x+y}{2}\right)\widetilde{v}(\xi) - \gamma_a(x)\,\widetilde{v}(\xi)
\end{aligned}$$

and

$$\begin{aligned}
r_2(x,y,\xi) &= a_2(x,y,\xi) - a_2(x,x,\xi) \\
&= c(x,y)\,\gamma_a(x)\,\widetilde{v}(\xi) - \gamma_a(x)\,\widetilde{v}(\xi).
\end{aligned}$$

The compactness of each of the last pseudodifferential operators easily follows from Theorem 2.3, Theorems 4.1 - 4.4, and the fact that $\widetilde{v}(\xi)$ is a C^∞-function with a compact support. Indeed, the above property of $\widetilde{v}(\xi)$ guarantees that both $a_1(x,y,\xi)$ and $a_2(x,y,\xi)$, as well as their two consecutive derivatives on ξ satisfy the second property in (2.6); while the properties

$$\lim_{(x,y)\to\infty} \frac{\partial^k d_{1,2}}{\partial y^k}(x,y) = 0, \qquad \text{for} \quad k = 1,2,$$

where $d_1(x,y) = c(x,y)\,\gamma_a\left(\frac{x+y}{2}\right)$ and $d_2(x,y) = c(x,y)\,\gamma_a(x)$ imply the first equqlity in (2.6). □

The above result leads directly to the following extension (of the sufficient part) of Theorem 1.3.

Corollary 2.6 *Let the operator $A \in \mathcal{T}(PC(\overline{\mathbb{D}},T))$ be such that in its canonical representation*

$$A = T_{s_A} + \sum_{k=1}^{m} T_{v_k} f_{A,k}(T_{\chi_k})T_{v_k} + K$$

all operators $f_{A,k}(T_{\chi_k})$ are Toeplitz with possibly unbounded symbols a_k, $k = 1, ..., m$, correspondingly. Then $A = T_a + K_A$ is a compact perturbation of the Toeplitz operator T_a, where

$$a(z) = s_A(z) + \sum_{k=1}^{m} a_k(z) v_k^2(z),$$

where $s_A(z)$ is given by (1.3).

We note that Corollary 2.6 immediately unhides via property (1.5) many Toeplitz operators in $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$ having unbounded symbols. Indeed, recall in this connection the following result ([4, Theorem 6.2]).

For any L_1-symbol $a(\theta) \in H(L_1(0, \pi))$ we define the following averaging functions, corresponding to the endpoints of $[0, \pi]$,

$$C_a^{(1)}(\theta) = \int_0^\theta a(u) du, \qquad D_a^{(1)}(\theta) = \int_{\pi-\theta}^{\pi} a(u) du$$

and

$$C_a^{(p)}(\theta) = \int_0^\theta C_a^{(p-1)}(u) du, \qquad D_a^{(p)}(\theta) = \int_{\pi-\theta}^{\pi} D_a^{(p-1)}(u) du,$$

for each $p = 2, 3, ...$

Next statement gives the conditions on some regular behavior of L_1-symbols near endpoints 0 and π guaranteeing that the corresponding Toeplitz operators is a certain continuous function of T_{χ_+}, and thus belong to the algebra $\mathcal{T}_+$.

Theorem 2.7 *Let $a(\theta) \in H(L_1(0, \pi))$ and for some $p, q \in \mathbb{N}$,*

$$\lim_{\theta \to 0} \theta^{-p} C_a^{(p)}(\theta) = c_p \ (\in \mathbb{C}) \qquad \text{and} \qquad \lim_{\theta \to \pi} \theta^{-q} D_a^{(q)}(\theta) = d_q \ (\in \mathbb{C}). \tag{2.8}$$

Then $\gamma_a(\lambda) \in C(\overline{\mathbb{R}})$, and thus $T_a \in \mathcal{T}_+$.

The conditions (2.8) are obviously satisfied, with $p = q = 1$, for example, for any function $a(\theta) \in H(L_1(0, \pi))$ which has limits at the endpoints of $[0, \pi]$. Of course, the existence of symbol limits at the endpoints by no means is necessary for the Toeplitz operator T_a to be an element of $\mathcal{T}_+$. As Example 1.5 shows, the corresponding symbol can even be unbounded near each of the endpoints 0 and π. Many further particular symbols can be given, for example, by combining polynomial growth with logarithmic and itterated logarithmic growth, then by considering linear combinations of different symbols, etc. The following symbol may serve as an illustrative example,

$$a(\theta) = \sum_{k=1}^{n} c_k \theta^{-\beta_k} \ln^{\lambda_k} \theta^{-1} \sin\left(\theta^{-\alpha_k} \ln^{\mu_k} \theta^{-1}\right),$$

where $c_k \in \mathbb{C}$, $0 < \beta_k < 1$, $\alpha_k > 0$, $\lambda_k \in \mathbb{R}$, $\mu_k \in \mathbb{R}$, $k = 1, ..., n$.

We mention especially that when speaking about a compact perturbation of a Toeplitz operator, say T_a, one should always remember that the coset $T_a + \mathcal{K}$ contains many Toeplitz operators of the form T_{a+k} for which the Toeplitz operator T_k is compact; and that all such operators have the same image $\operatorname{sym} T_{a+k} = \operatorname{sym} T_a$ in the (Fredholm) symbol algebra $\operatorname{Sym} \mathcal{T}(PC(\overline{\mathbb{D}}, T))$. At the same time the properties of the functions a and $a + k$ can be extremely different. Indeed, even having as nice as possible a, say $a \in C(\overline{\mathbb{D}})$, one can always add, for example, the function

$$k(z) = (1 - r^2)^{-\beta} \sin(1 - r^2)^{-\alpha} + (1 - r)\chi_Q(z), \qquad z = re^{i\theta},$$

where the first summand is taken from Example 2.1 and Q is the set of all points $z = r_1 + ir_2 \in \overline{\mathbb{D}}$ with rational r_1 and r_2. This converts the initial symbol a to the symbol $a + k$, which does not have a limit at every point of $\overline{\mathbb{D}}$, and moreover is unbounded near every point of the boundary.

That is, whenspeaking about the representation $A = T_a + K$ it is preferable to have a symbol a with less unnecessary singularities. It seems that the option given by Theorem 1.4 and Corollary 2.6 may be optimal in this respect.

3 Toeplitz or not Toeplitz

The key question in the description of Toeplitz operators in $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$ is whether the operators of the form $f(T_{\chi_k})$, where $f(x) \in C[0, 1]$ and $k = 1, 2, ..., m$, are Toeplitz or not. By (1.5) this question is reduces to the following question in the upper half-plane setting: given $f(x) \in C[0, 1]$, whether the operator $f(T_{\chi_+})$ is Toeplitz or not. The last questions is in turn equivalent to: whether the function $\gamma(\lambda) \in C(\overline{\mathbb{R}})$, which is connected with $f(x) \in C[0, 1]$ by (see (1.7))

$$\gamma(\lambda) = f\left(\frac{1}{e^{-\pi\lambda} + 1}\right),$$

admits the representation (1.6) for some $a(\theta) \in L_1(0, \pi)$, i.e.,

$$\gamma(\lambda) = \gamma_a(\lambda) = \frac{2\lambda}{1 - e^{-2\pi\lambda}} \int_0^{\pi} a(\theta)\, e^{-2\lambda\theta}\, d\theta, \qquad \lambda \in \overline{\mathbb{R}}. \tag{3.1}$$

The statements of the next theorem are necessary for the existence of the above representation for a given function $\gamma(\lambda) \in C(\overline{\mathbb{R}})$.

Theorem 3.1 *Let $a(\theta) \in L_1(0,\pi)$. Then the function $\gamma_a(\lambda)$ is analytic in the whole complex plane with the exception of the points $\lambda_n = in$, where $n = \pm 1, \pm 2, \ldots$, where $\gamma_a(\lambda)$ has simple poles. Moreover, for any fixed and sufficiently small δ the function $\gamma_a(\lambda)$ admits on the set*

$$\mathbb{C} \setminus \bigcup_{\mathbb{Z}\setminus\{0\}} K_n(\delta), \qquad \text{where} \qquad K_n(\delta) = \{\lambda \in \mathbb{C} : \ |\lambda - in| < \delta\},$$

the following estimate

$$|\gamma_a(\lambda)| \le \text{const}\, |\lambda|,$$

where const *depends on δ.*

Proof. The function

$$\beta_a(\lambda) = \int_0^\pi a(\theta)\, e^{-2\lambda\theta}\, d\theta, \qquad \lambda = x + iy,$$

is obviously analytic in $\mathbb{C}$, and for large $|\lambda|$ admits the estimate

$$|\beta_a(\lambda)| \le \int_0^\pi |a(\theta)|\, e^{-2x\theta}\, d\theta.$$

Thus for $x > 0$ we have

$$|\beta_a(\lambda)| \le \|a(\theta)\|_{L_1},$$

while for $x < 0$ we have

$$\begin{aligned} |\beta_a(\lambda)| &\le e^{-2\pi x} \int_0^\pi |a(\theta)|\, e^{2x(\pi-\theta)}\, d\theta \\ &= e^{-2\pi x} \int_0^\pi |a(\theta)|\, d\theta \le e^{-2\pi x}\, \|a(\theta)\|_{L_1}. \end{aligned}$$

The theorem statements now follow from

$$\gamma_a(\lambda) = \frac{2\lambda}{1 - e^{-2\pi\lambda}}\, \beta_a(\lambda).$$

□

To give a sufficient condition for the representation (3.1) we start with some definitions (see [1] for details).

An entire function $\varphi(\lambda)$ is called a function of exponential type if it obeys an estimate

$$|\varphi(\lambda)| \leq A e^{B|\lambda|},$$

where the positive constants A and B do not depend on $\lambda \in \mathbb{C}$. The infimum of all constants B for which this estimate holds is called the type of the function $\varphi(\lambda)$.

We denote by $\mathcal{L}_2^\sigma$ the set of all functions of exponential type less than or equal than σ whose restrictions to $\mathbb{R}$ belong to $L_2(\mathbb{R})$.

An analytic function on the upper half-plane $\varphi(\lambda)$ is said to belong to the Hardy space $H^2(\mathbb{R})$ if

$$\sup_{y>0} \int_{\mathbb{R}} |\varphi(x+iy)|^2 dx < \infty.$$

The proof of the next theorem can be found, for example, in [1, Theorem 1.4].

Theorem 3.2 *Let $\varphi(z) \in \mathcal{L}_2^{2\pi} \cap H^2(\mathbb{R})$. Then there exists a function $a(\theta) \in L_2(0, 2\pi)$ such that*

$$\varphi(z) = \int_0^{2\pi} a(\theta)\, e^{iz\theta}\, d\theta, \qquad \lambda \in \mathbb{C}.$$

As $L_2(0, 2\pi) \subset L_1(0, 2\pi)$, the theorem can be used as a sufficient condition for the existence of representation (3.1). Indeed, given a function $\gamma(\lambda)$, introduce

$$\varphi(z) = i \frac{1 - e^{i\pi z}}{z}\, \gamma\left(-\frac{iz}{2}\right).$$

If this function $\varphi(z)$ belongs to $\mathcal{L}_2^{2\pi} \cap H^2(\mathbb{R})$ then $\gamma(\lambda)$ does admit representation (3.1). That is, there a function $a(\theta) \in L_1(0, 2\pi)$ such that $\gamma(\lambda) = \gamma_a(\lambda)$ and

$$T_a = R^* \gamma(\lambda) R = f(T_{\chi_+}),$$

where

$$f(x) = \gamma\left(\gamma_{\chi_+}^{-1}(x)\right) = \gamma\left(-\frac{1}{\pi} \ln \frac{1-x}{x}\right).$$

Theorem 3.3 *Let*

$$p(x) = \sum_{k=1}^{n} a_k x^k, \qquad a_n \neq 0,$$

be a polynomial of degree $n \geq 2$ with complex coefficients. Then the bounded operator $p(T_{\chi_+})$ is not a Toeplitz operator.

PROOF. The operator $p(T_{\chi_+})$ belongs to the algebra generated by all Toeplitz operators on the upper half-plane with homogeneous L_∞-symbols $a(\theta)$ of zero order. Thus by [2] the operator $p(T_{\chi_+})$ being Toeplitz must have a symbol which belongs to $H(L_1(0,\pi))$. The corresponding function $\gamma(\lambda)$, that is, such that $p(T_{\chi_+}) = R^*\gamma(\lambda)R$, obviously has the form

$$\gamma(\lambda) = p\left(\gamma_{\chi_+}(\lambda)\right) = p\left(\frac{1}{e^{-\pi\lambda}+1}\right).$$

But this function has poles of order n at the points $\lambda_n = i(2n-1)$, where $n \in \mathbb{Z}$. Thus by Theorem 3.1 there is no function $a(\theta) \in H(L_1(0,\pi))$ for which the representation (3.1) holds. □

Corollary 3.4 *Let A be an operator of the algebra $\mathcal{T}(PC(\overline{\mathbb{D}},T))$ having the form*

$$A = \sum_{i=1}^{p}\prod_{j=1}^{q_i} T_{a_{i,j}},$$

where all $a_{i,j} \in PC(\overline{\mathbb{D}},T)$. Then A is a compact perturbation of a Toeplitz operator if and only if A is a compact perturbation of one of the initial generators of $\mathcal{T}(PC(\overline{\mathbb{D}},T))$, which is a Toeplitz operator T_a with $a \in PC(\overline{\mathbb{D}},T)$.

PROOF. By Corollary 4.3 of [4], or Theorem 1.2 of this paper, the operator A admits the canonical representation

$$A = \sum_{i=1}^{p}\prod_{j=1}^{q_i} T_{a_{i,j}} = T_{s_A} + \sum_{k=1}^{m} T_{v_k} p_{A,k}(T_{\chi_k})T_{v_k} + K_A,$$

where $s_A = s_A(z) \in C(\overline{\mathbb{D}})$, $p_{A,k} = p_{A,k}(x)$, $k = 1,...,m$, are some polynomials, and K_A is a compact operator. Thus by Theorem 1.3, A is a compact perturbation of a Toeplitz operator if and only if each $p_{A,k}(T_{\chi_k})$, $k = 1,...,m$, is a Toeplitz operator, or by (1.5) if and only if each $p_{A,k}(T_{\chi_+})$, $k = 1,...,m$, is a Toeplitz operator. By Theorem 3.3 the last statement is equivalent to the fact that the degree of each polynomial $p_{A,k}(x)$, $k = 1,...,m$, must be lessthen or equal to one, which in turn is equivalent to the fact that A is a compact perturbation of a Toeplitz operator T_a with $a \in PC(\overline{\mathbb{D}},T)$. □

We summarize now the results obtained on Toeplitz operators of the algebra $\mathcal{T}(PC(\overline{\mathbb{D}},T))$. By its construction, the C^*-algebra $\mathcal{T}(PC(\overline{\mathbb{D}},T))$ consists of its initial generators, Toeplitz operators T_a with symbols $a \in PC(\overline{\mathbb{D}}, T)$, then of all elements of the form

$$\sum_{i=1}^{p} \prod_{j=1}^{q_i} T_{a_{i,j}},$$

forming thus a nonclosed algebra, and finally of all elements of the uniform closure of the nonclosed algebra. The information on Toeplitz operators is as follows.

- All initial generators are Toeplitz operators.
- None of the elements of the nonclosed algebra which does not reduce to a compact perturbation of an initial generator can be (a compact perturbation of) a Toeplitz operator. Thus at this stage we have not increased the quantity of Toeplitz operators.
- The uniform closure of the nonclosed algebra contains a huge amount of Toeplitz operators, with bounded and even unbounded symbols, which are drastically different from the initial generators. All these Toeplitz operators are uniform limits of sequences of non-Toeplitz operators.
- The uniform closure, apart of Toeplitz operators, contains much more non-Toeplitz operators (this is a consequence of Theorem 3.1).

At the same time each operator in the C^*-algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$ admits a very transparent canonical representation (given in Theorem 1.2).

4 Appendix: Technical statements

We prove here of several statements whose results were used in Theorems 2.4 and 2.5.

We start with some properties of the function (see (1.9))

$$c(x, y) = \frac{1 - e^{-\pi(x+y)}}{x + y} \sqrt{\frac{2x}{1 - e^{-2\pi x}}} \sqrt{\frac{2y}{1 - e^{-2\pi y}}}, \qquad x, y \in \mathbb{R}.$$

Theorem 4.1 *The function $c(x, y)$ is bounded in $\mathbb{R}^2$; i.e.,*

$$\sup_{(x,y)\in\mathbb{R}^2} |c(x, y)| < \infty.$$

PROOF. Introduce the function

$$f(u) = \sqrt{\frac{u}{1 - e^{-u}}}.$$

Then

$$c(x, y) = \frac{f(2\pi x)\, f(2\pi y)}{f^2(\pi(x + y))}.$$

Let $D_1 = [1, +\infty)$ and $D_{-1} = (-\infty, -1]$. We obviously have the following asymptotics in the above domains:

$$\begin{aligned}
f(u) &= u^{1/2} \left(1 + O(e^{-u})\right), & u \in D_1, \quad (4.1)\\
f(u) &= |u|^{1/2} e^{u/2} \left(1 + O(e^{u})\right), & u \in D_{-1}, \quad (4.2)\\
f^{-2}(u) &= u^{-1} \left(1 + O(e^{-u})\right), & u \in D_1, \quad (4.3)\\
f^{-2}(u) &= |u|^{-1} e^{-u} \left(1 + O(e^{u})\right), & u \in D_{-1}. \quad (4.4)
\end{aligned}$$

In what follows the relation $\varphi(u) \sim \psi(u)$ means that

$$0 < c \le \frac{\varphi(u)}{\psi(u)} \le C < \infty,$$

for all u in the domain under consideration. We note as well that if u belongs to any bounded domain in $\mathbb{R}$, then

$$f(u) \sim 1 \qquad \text{and} \qquad f^{-2}(u) \sim 1. \tag{4.5}$$

We will prove the statement of the theorem considering successively all possible locations of x and y on $\mathbb{R}$. The symmetry of $c(x, y)$ with respect to its arguments implies that it is sufficient to consider only the following cases:

1. x, $y \in [-1, 1]$. Then $x + y \in [-2, 2]$, and by (4.5) we have that $c(x, y) \sim 1$.

2. $x \in [-1, 1]$, $y \in D_1$. Then either $x + y \in D_1$ and thus by (4.2) and (4.4) we have

   $$c(x, y) \sim \frac{1 \cdot (2\pi y)^{1/2}}{\pi(x + y)} \sim y^{-1/2},$$

   or $y \in [1, 2]$ and thus, as in the first case, $c(x, y) \sim 1$.

3. $x \in [-1, 1]$, $y \in D_{-1}$. Then either $x + y \in D_{-1}$ and thus by (4.1) and (4.3) we have

   $$c(x, y) \sim \frac{1 \cdot |2\pi y|^{1/2} e^{\pi y}}{|\pi y| e^{\pi(x+y)}} \sim y^{-1/2},$$

   or $y \in [-2, -1]$ and again $c(x, y) \sim 1$.

4. $x \in D_1$, $y \in D_{-1}$. Then we have the following three possibilities for $x+y$:

   (a) $x+y \in [-1,1]$. Then by (4.1), (4.2), and (4.5), assuming that $x+y=\delta \in [-1,1]$, we have

$$c(x,y) \sim (2\pi x)^{1/2} \cdot (2\pi|y|)^{1/2} e^{\pi y} \cdot 1 \sim (\delta+y)^{1/2}|y|^{1/2}e^{\pi y} \sim |y|e^{\pi y}.$$

   (b) $x+y \in D_1$. Then by (4.1), (4.2), and (4.3) we have

$$\begin{aligned} c(x,y) &\sim \frac{(2\pi x)^{1/2} \cdot (2\pi|y|)^{1/2} e^{\pi y}}{\pi(x+y)} \sim \begin{cases} \frac{x^{1/2}|y|^{1/2}e^{\pi y}}{x}, & x \geq 2|y| \\ \frac{|y|^{1/2}|y|^{1/2}e^{\pi y}}{1}, & x < 2|y| \end{cases} \\ &\sim \begin{cases} \frac{|y|^{1/2}e^{\pi y}}{x^{1/2}}, & x \geq 2|y| \\ |y|e^{\pi y}, & x < 2|y| \end{cases} . \end{aligned}$$

   (c) $x+y \in D_{-1}$. Then by (4.1), (4.2), and (4.4) we have

$$\begin{aligned} c(x,y) &\sim \frac{(2\pi x)^{1/2} \cdot (2\pi|y|)^{1/2} e^{\pi y}}{\pi|x+y|e^{\pi(x+y)}} \sim \frac{x^{1/2}|y|^{1/2}e^{-\pi x}}{|x+y|} \\ &\sim \begin{cases} \frac{|x|^{1/2}e^{-\pi x}}{y^{1/2}}, & |y| \geq 2x \\ xe^{-\pi x}, & |y| < 2x \end{cases} . \end{aligned}$$

5. $x \in D_1$, $y \in D_1$. Then $x+y \in D_1$ and thus by (4.1) and (4.3) we have

$$\begin{aligned} c(x,y) &= \frac{(2\pi x)^{1/2}(2\pi y)^{1/2}}{\pi(x+y)} \left(1+O(e^{-x})+O(e^{-y})\right) \\ &= \frac{2x^{1/2}y^{1/2}}{x+y} \left(1+O(e^{-x})+O(e^{-y})\right). \end{aligned}$$

   As $2x^{1/2}y^{1/2} \leq x+y$, the boundedness of $c(x,y)$ is obvious.

6. $x \in D_{-1}$, $y \in D_{-1}$. Then $x+y \in D_{-1}$ and thus by (4.2) and (4.4) we have

$$\begin{aligned} c(x,y) &= \frac{(2\pi|x|)^{1/2}e^{\pi x}(2\pi|y|)^{1/2}e^{\pi y}}{\pi|x+y|e^{\pi(x+y)}} \left(1+O(e^{-|x|})+O(e^{-|y|})\right) \\ &= \frac{2|x|^{1/2}|y|^{1/2}}{|x|+|y|} \left(1+O(e^{-|x|})+O(e^{-|y|})\right). \end{aligned}$$

The theorem is proved. $\square$

Theorem 4.2 *Both functions* $\frac{\partial c}{\partial x}(x,y)$ *and* $\frac{\partial c}{\partial y}(x,y)$ *are bounded in* $\mathbb{R}^2$*, and moreover*

$$\lim_{(x,y)\to\infty} \frac{\partial c}{\partial x}(x,y) = 0 \qquad \text{and} \qquad \lim_{(x,y)\to\infty} \frac{\partial c}{\partial y}(x,y). \tag{4.6}$$

PROOF. We start with the asymptotics of the derivatives of $f(u)$. We have

$$f'(u) = \frac{1}{2}\sqrt{\frac{1-e^{-u}}{u}} \cdot \frac{1-e^{-u}+ue^{-u}}{(1-e^{-u})^2},$$

thus, as is easy to see,

$$\begin{aligned} f'(u) &= u^{-1/2}\left(1+O(ue^{-u})\right), \qquad & u \in D_1, & \qquad (4.7)\\ f'(u) &= |u|^{1/2}e^{u/2}\left(1+O(e^{u})\right), \qquad & u \in D_{-1}. & \qquad (4.8) \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial c}{\partial y}(x,y) &= f(2\pi x) \cdot \frac{2\pi f'(2\pi y) f^2(\pi(x+y)) - 2\pi f(2\pi y) f(\pi(x+y)) f'(\pi(x+y))}{f^4(\pi(x+y))} \\ &= 2\pi\, c(x,y) \left(\frac{f'(2\pi y)}{f(2\pi y)} - \frac{f'(\pi(x+y))}{f(\pi(x+y))}\right). \end{aligned} \tag{4.9}$$

We check now the boundedness of the logarithmic derivative of f. By (4.1) and (4.7) we have

$$\begin{aligned} \frac{f'(2\pi y)}{f(2\pi y)} &= \frac{(2\pi y)^{-1/2}}{(2\pi y)^{1/2}}\left(1+O(ye^{-2\pi y})\right) \\ &= (2\pi y)^{-1}\left(1+O(ye^{-2\pi y})\right), \qquad y \in D_1, \end{aligned}$$

and by (4.2) and (4.8),

$$\begin{aligned} \frac{f'(2\pi y)}{f(2\pi y)} &= \frac{(2\pi |y|)^{1/2}e^{\pi y}}{(2\pi |y|)^{1/2}e^{\pi y}}\left(1+O(e^{2\pi y})\right) \\ &= 1+O(e^{2\pi y}), \qquad y \in D_{-1}. \end{aligned}$$

Thus the function $\frac{\partial c}{\partial y}(x,y)$ is bounded in $\mathbb{R}^2$.

To prove the second equality in (4.6) we note that if both $|y| \to \infty$, and $|x+y| \to \infty$, and moreover $\operatorname{sign} y = \operatorname{sign}(x+y)$ then the result follows from (4.9) and boundedness of $c(x,y)$. If both $|y| \to \infty$, and $|x+y| \to \infty$, but $\operatorname{sign} y = -\operatorname{sign}(x+y)$ then by case 4.b of Theorem 4.1 we have that $c(x,y) \to 0$. If $(x,y) \to \infty$ while y belongs to a bounded domain, then $x+y$ is unbounded and we are in the situation of the cases 2 or 3 of Theorem 4.1, when $c(x,y) \to 0$. Finally, if $(x,y) \to \infty$ but $x+y$ is bounded, then as in the case 4.a of Theorem 4.1 we have that $c(x,y) \to 0$. In the last three cases the result follows from (4.9), boundedness of the logarithmic derivates, and $c(x,y) \to 0$.

Boundedness of $\frac{\partial c}{\partial x}(x,y)$ and the first equality in (4.6) follow from the above and the symmetry of $c(x,y)$ with respect to x and y. $\square$

Theorem 4.3 *The function $\frac{\partial^2 c}{\partial y^2}(x,y)$ is bounded in $\mathbb{R}^2$, and moreover*

$$\lim_{(x,y)\to\infty} \frac{\partial^2 c}{\partial y^2}(x,y) = 0.$$

PROOF. For the second derivative of f, after elementary calculations, we have

$$f''(u) = u^{-3/2}\left(1 + O(ue^{-u})\right), \qquad u \in D_1, \tag{4.10}$$

$$f''(u) = |u|^{1/2} e^{u/2} \left(1 + O(e^{u})\right), \qquad u \in D_{-1}. \tag{4.11}$$

Differentiating (4.9) we have

$$\begin{aligned}
\frac{\partial^2 c}{\partial y^2}(x,y) &= 2\pi \frac{\partial c}{\partial y}(x,y) \left(\frac{f'(2\pi y)}{f(2\pi y)} - \frac{f'(\pi(x+y))}{f(\pi(x+y))} \right) \\
&- 2\pi^2 c(x,y) \left[2 \left(\frac{f''(2\pi y)}{f(2\pi y)} - \left(\frac{f'(2\pi y)}{f(2\pi y)} \right)^2 \right) \right. \\
&- \left. \left(\frac{f''(\pi(x+y))}{f(\pi(x+y))} - \left(\frac{f'(\pi(x+y))}{f(\pi(x+y))} \right)^2 \right) \right].
\end{aligned}$$

We note that by Theorem 4.2 the first summand is bounded and tends to 0 as $(x,y) \to \infty$. Considering the second summand we have that if $|x+y| \to \infty$ and $y \to \infty$ then formulas (4.1), (4.2), (4.7), (4.8), (4.10), and (4.11), for $u = 2\pi y$ or $u = \pi(x+y)$, yield

$$\begin{aligned}
\frac{f''(u)}{f(u)} - \left(\frac{f'(u)}{f(u)} \right)^2 &= \begin{cases} u^{-2}\left(1 + O(ue^{-u})\right) - u^{-2}\left(1 + O(ue^{-u})\right), & u \in D_1 \\ \left(1 + O(e^{u})\right) - \left(1 + O(e^{u})\right), & u \in D_{-1} \end{cases} \\
&= \begin{cases} O(u^{-1}e^{-u}), & u \in D_1 \\ O(e^{u}), & u \in D_{-1} \end{cases}.
\end{aligned}$$

If $(x,y)\to\infty$, but $|x+y|$ is bounded, then we are in the situation of the case 4.a of Theorem 4.1, and thus $c(x,y)\to 0$ as $(x,y)\to\infty$. Finally, if $(x,y)\to\infty$, but y is bounded, then we are in the situation of the cases 2 or 3 of Theorem 4.1, and thus $c(x,y)\to 0$ as $(x,y)\to\infty$. □

Theorem 4.4 *Let $a(\theta)\in L_1(0,\pi)$ be such that*

$$\gamma_a(\lambda)=\frac{2\lambda}{1-e^{-2\pi\lambda}}\int_0^\pi a(\theta)e^{-2\lambda\theta}\,d\theta\in L_\infty(\mathbb{R}).$$

Then, for each $k=1,2,...,$

$$\lim_{\lambda\to\pm\infty}\frac{d^j\gamma_a}{d\lambda^j}(\lambda)=0.$$

PROOF. Let $k=1$. Then

$$\begin{aligned}\frac{d\gamma_a}{d\lambda}(\lambda) &= \frac{2}{1-e^{-2\pi\lambda}}\int_0^\pi a(\theta)e^{-2\lambda\theta}\,d\theta+\frac{4\pi\lambda e^{-2\pi\lambda}}{(1-e^{-2\pi\lambda})^2}\int_0^\pi a(\theta)e^{-2\lambda\theta}\,d\theta\\ &- \frac{4\lambda}{1-e^{-2\pi\lambda}}\int_0^\pi \theta a(\theta)e^{-2\lambda\theta}\,d\theta=I_1(\lambda)+I_2(\lambda)+I_3(\lambda).\end{aligned}$$

We consider first the behaviour of the derivative when $\lambda\to+\infty$. We have

$$I_1(\lambda)=\frac{2}{\lambda}\,\gamma_a(\lambda)\qquad\text{and}\qquad I_2(\lambda)=\frac{2\pi e^{-2\pi\lambda}}{1-e^{-2\pi\lambda}}\,\gamma_a(\lambda),$$

and thus the first two summands tend to 0 as $\lambda\to+\infty$.

Consider now the last summand,

$$\begin{aligned}|I_3(\lambda)| &\le 2\int_0^\delta |a(\theta)|(2\lambda\theta)e^{-2\lambda\theta}d\lambda+4\pi\lambda e^{-2\lambda\delta}\int_\delta^\pi |a(\theta)|d\theta\\ &\le \int_0^\delta |a(\theta)|d\theta+4\pi\lambda e^{-2\lambda\delta}\int_0^\pi |a(\theta)|d\theta.\end{aligned}$$

Then for any $\varepsilon>0$ we can select both δ small enough and $\lambda_0=\lambda_0(\delta)$ large enough, such that

$$\int_0^\delta |a(\theta)|d\theta<\varepsilon/2$$

and

$$4\pi\lambda e^{-2\lambda\delta} \int_0^\pi |a(\theta)|d\theta < \varepsilon/2,$$

for all $\lambda \geq \lambda_0(\delta)$. That is, $\lim_{\lambda\to+\infty} I_3(\lambda) = 0$.

The case when $\lambda \to -\infty$ follows from the above and the equality

$$\gamma_{a(\theta)}(\lambda) = \gamma_{a(\pi-\theta)}(-\lambda).$$

The cases when $k > 1$ are considered analogously. □

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