Double Barrier Options Under Lévy Processes

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To I. B. Simonenko on the occasion of his 70th birthday

Abstract. In this paper the problem of determination of the no arbitrage price of double barrier options in the case of stock prices is modelled on Lévy processes is considered. Under the assumption of existence of the Equivalent Martingale Measure this problem is reduced to the convolution equation on a finite interval with symbol generated by the characteristic function of the Lévy process. We work out a theory of unique solvability of the getting equation and stability of the solution under relatively small perturbations.

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1. Introduction

The problem of the determination of the price of a double barrier option in case when the stock price is modelled by geometric Brownian motion (classical hypothesis) is considered in [1]–[8]. The articles [4]–[8] are devoted to an approach connected with a solution of the Black-Scholes (partial) differential equation on a strip of finite width. But it should be noted that for many cases geometric Brownian motion is not an adequate model for stock price. Therefore in recent years many investigators have used Lévy processes as models for logarithmic stock price. In this way European options ([9]–[19]), perpetual American options ([15], [20]–[21]) and barrier options ([15], [22], [23]) were considered.

In this paper we consider double barrier options under Lévy processes. Following the monograph [15] we use the generalized Black-Scholes equation approach.

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That is we reduce the original option problem to a partial pseudodifferential equation of the type

$$\frac{\partial u(x,t)}{\partial t} - L_x u(x,t) = 0 \tag{1}$$

where the pseudodifferential, more exactly, convolution operator L_x (acting on the variable x) is generated by the characteristic exponent of the Lévy process $X_t := \ln S_t$ (here S_t is stock price). In the case of a double barrier the equation (1) is considered in the region

$$E = (x_1, x_2) \times (-\infty, T)$$

(where T is expiry date) and can be reduced (with the help of the Laplace transform in the variable $\tau = T - t$) to a convolution equation on an interval (x_1, x_2). We apply the Matrix Riemann Boundary Value Problem method worked out in the papers [24]–[27] for the investigation of the convolution equation. In this way we prove unique solvability of the problem and stability of the solution under relatively small perturbations.

2. Auxillary material

In this section we introduce necessary definitions and formulate some well known results (see [15], [28], [29]) There are many kinds of double barrier option problems. We consider (in some sense) the basic problem which can be called Up-Down-And-Out barrier option. Other double barrier option problems can be reduced to this problem and (or) to single barrier problems.

Let S_t be stock price at the instant of time t, and $\varphi : (0, \infty) \to [0, \infty)$ be a measurable function.

Definition 2.1. *An Up-Down-And-Out barrier option is an agreement between two persons (Writer and Holder) at time instant t according to which Writer is obliged to pay to Holder the amount $\varphi(S_T)$ at the future instant of time T (expiry date) if and only if during the option life (between t and T), S_t is always within the interval (S_1, S_2) (here $0 < S_1 < S_2$ are some levels, i.e., barriers, of the stock price).*

Note that if there exists some instant of time $t_{1,2} \leq T$ such that $S_{t_1} \geq S_2$ or $S_{t_2} \leq S_1$ then the option expires worthless.

Consider a market model which consists of a bond with a constant riskless rate of return $r > 0$, and of a stock with price $S_t = \exp\{X_t\}$ where X_t is a Lévy process.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space where Ω is the space of elementary events and $\mathcal{F}$ is a σ-algebra of subsets of Ω.

Definition 2.2 ([28]–[29])**.** *An $\mathcal{F}$-adapted process X_t is called a Lévy process if the following conditions hold:*

1. $X_0 = 0$ *a.e.*

2. *X_t has stationary increment, that is, for arbitrary $t > s > 0$ the distribution of $(X_t - X_s)$ coincides with the distribution of X_{t-s}.*
3. *X_t has independent increments, that is, for arbitrary $0 \le t_1 < t_2 < \ldots < t_n$, the random variables*
$$X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$$
are independent.
4. *For each $w \in \Omega$ the function $X_t = X_t(w)$ is right-continuous on $(0, \infty)$ and there exists a left limit at all $t \ge 0$.*
5. *X_t is stochastically continuous, that is for every $t > 0$ and $\varepsilon > 0$*
$$\lim_{s \to t} \mathbf{P}[|X_t - X_s| > \varepsilon] = 0.$$

If X_t is a Lévy process, then according to the Lévy-Khintchine formula ([28]–[29])

$$E_{\mathbf{P}}[e^{i\xi X_t}] = e^{-t\psi^{\mathbf{P}}(\xi)}, \quad \xi \in \mathbb{R}, \tag{1}$$

where the function $\psi^{\mathbf{P}}(\xi)$ has the representation

$$\psi^{\mathbf{P}}(\xi) = \frac{1}{2}\sigma^2\xi^2 - i\mu\xi - \int_{-\infty}^{\infty} (e^{iu\xi} - 1 - i\xi u I_{(-1,1)}(u))\Pi(du) \tag{2}$$

with $\sigma \ge 0$, $\mu \in \mathbb{R}$, and Π is a measure on $\mathbb{R}$ satisfying the condition

$$\left| \int_{-\infty}^{\infty} \frac{u^2}{1+u^2} \Pi(du) \right| < \infty, \tag{3}$$

$$I_{(-1,1)}(u) = \begin{cases} 1, & |u| < 1; \\ 0, & |u| \ge 1. \end{cases}$$

The expectation of exponent $E_{\mathbf{P}}[e^{i\xi X_t}]$ is called the characteristic function, the function $\psi^{\mathbf{P}}(\xi)$ is called the characteristic exponent of X_t (under the probability measure $\mathbf{P}$), the triplet (a, γ, Π) is called the generating triplet of X_t.

We will consider an arbitrage free market (see,for example, [30]). From results of [31] it follows that no-arbitrage pricing of options is possible if there exists an Equivalent Martingale Measure (EMM) $\mathbf{Q}$.

Let (Ω, F, F_t, P) be a probability space with right continuous filtration $F_t(\subset F)$.

Let $\mathbf{P}_t$, $\mathbf{Q}_t$ be restriction measures $\mathbf{P}\Big|_{\mathcal{F}_t}$ and $\mathbf{Q}\Big|_{\mathcal{F}_t}$ respectively. Let $Z_t = \frac{d\mathbf{Q}_t}{d\mathbf{P}_t}$ be the density of $\mathbf{Q}_t$ with respect to $\mathbf{P}_t$. If $0 < Z_t < \infty$ a.e. then the measures $\mathbf{P}$ and $\mathbf{Q}$ are called locally equivalent.

Definition 2.3. *A measure $\mathbf{Q}$ locally equivalent with respect to the measure $\mathbf{P}$ is called an Equivalent Martingale Measure (EMM) if the process $S_t^* = e^{-rt}S_t$ is a $\mathbf{Q}$-martingale (more exactly: is a $(\Omega, \mathcal{F}_t, \mathbf{Q})$-martingale).*

The process S_t^* is called the discounted stock price.

Let $V(S_t, t)$ be the option price for time $t \le T$, and let $V^*(S_t, t) := e^{-rt}V(S_t, t)$ be discounted option price. Under the measure $\mathbf{Q}$ all discounted price processes (such that the prices are $\mathbf{Q}$-integrable) are assumed to be martingales. By virtue of this assumption prices of certain securities whose price at some future date T are given random walk can be expressed by the help of conditional expectation. We write the conditional expectation formula for our case.

Let $x_{1,2} = \ln S_{1,2}$ and let $\eta := \eta(w)$ be the hitting time of the set $\mathbb{R} \setminus (x_1, x_2)$

$$\eta(w) := \inf\{t \geq 0 | X_t(w) \in \mathbb{R} \setminus (x_1, x_2)\}.$$

Then for the Up-Down-And-Out barrier option at expiry date $t = T$ we have

$$V(e^{X_T}, T) = \varphi(e^{X_T}) I_{\{\eta > T\}}$$

where I_A is the characteristic function of the set $A \subset \Omega$. Denote $\mathcal{U}(X_t, t) := V(e^{X_t}, t)$ and $g(X_T) = \varphi(e^{X_T})$. Then we have

$$e^{-rt}\mathcal{U}(x,t) = E_{\mathbf{Q}} \left[e^{-rT} g(X_T) I_{\{\eta > T\}} \Big|_{\mathcal{F}_t} \right] \tag{4}$$

where the right-hand side is the conditional expectation under the measure $\mathbf{Q}$ with respect to the σ-algebra $\mathcal{F}_t$ with $X_t = x$.

Thus the existence of EMM is an important question in option theory. If X_t is neither Brownian motion nor a Poisson process then typically that EMM is not unique. Moreover there often exist infinitely many different EMMs. We formulate in this connection the main result of the article [32].

Suppose that X_t is a Lévy process with characteristic triplet $(0, \mu, \Pi)$ (for a similar result for processes with triplet (σ, μ, Π) for $\sigma > 0$ see [9]).

Let μ_r denote the class of measures $\mathbf{Q}$ locally equivalent to $\mathbf{P}$ under which $e^{-rt}S_t$ is a martingale and X_t is a Lévy process under the measure $\mathbf{Q}$.

Let $Y_{\mu,r}(\Pi(dx))$ denote the class of function $y : \mathbb{R} \to (0, +\infty)$ such that

$$\int_{-\infty}^{\infty} (\sqrt{y(x)} - 1)^2 \Pi(dx) + \int_{\{x>1\}} (e^x - 1) y(x) \Pi(dx) < \infty$$

and

$$\mu - r + \int_{-\infty}^{\infty} ((e^x - 1) y(x) - x I_{[-1,1]}(x)) \Pi(dx) = 0.$$

Theorem 2.1 ([32]). a) *If $Y_{\mu,r}(\Pi(dx)) = \emptyset$, then $\mu_r = \emptyset$.*

b) *If $Y_{\mu,r}(\Pi(dx)) \neq \emptyset$, then μ_r is non-empty and for each $y \in Y_{\mu,r}(\Pi(dx))$ there is a measure $\mathbf{Q} \in \mu_r$ under which X_t is again a Lévy process with generating triplet $(0, \mu', \Pi')$, where*

$$\mu' = \mu + \int_{-1}^{1} x(y(x) - 1) \Pi(dx)$$

and

$$\Pi'(A) = \int_A y(x) \Pi(dx).$$

Conversely, if $\mathbf{Q} \in \mu_r$ *is the measure under which* X_t *is a Lévy process, then its generating triplet is* $(0, \mu', \Pi')$ *where* μ' *and* Π' *are given by the above expressions with some* $y \in Y_{\mu,r}$.

c) *Let* y *and* $\mathbf{Q}$ *be as in b). Then the characteristic exponents of* X_t *under* $\mathbf{P}$ *and* $\mathbf{Q}$ *are related by*

$$\psi^{\mathbf{Q}}(\xi) = \psi^{\mathbf{P}}(\xi) + \int_{-\infty}^{\infty} (1 - e^{ix\xi})(y(x) - 1)\Pi(dx).$$

Thus in the case of Lévy processes typically an EMM exists and is not unique. Thus there exists the problem of the choice of EMM. This problem is not trivial. For a discussion about such choice, see [15] (pp. 97-98).

From now on assume that an EMM $\mathbf{Q}$ is chosen and that X_t is a Lévy process under the measure $\mathbf{Q}$.

Definition 2.4 We will say that the Lévy process X_t satisfies the (ACP)-condition (see [15] p. 59) if the function

$$(U^r f)(x) := E_{\mathbf{Q}}[\int_0^{\infty} e^{-rt} f(X_t) dt | X_0 = x]$$

is continuous for every $f \in L_\infty(R)$.

Some sufficient conditions for (ACP)-condition are given (for example) in [15] (Theorem 2.11 and Lemma 2.4).

We will consider everywhere below Lévy processes that satisfy the (ACP)-condition.

Let $g \in L_\infty(x_1, x_2)$, the set of all essentially bounded functions on (x_1, x_2) and let the process X_t satisfy the (ACP)-condition. Then according to Theorem 2.13 of [15] the function $\mathcal{U}(x,t)$ defined by (4) is a bounded solution of the following partial pseudodifferential problem:

$$\frac{\partial \mathcal{U}(x,t)}{\partial t} - (r - L_x^{\mathbf{Q}})\mathcal{U}(x,t) = 0, \ x \in (x_1, x_2), \ t < T, \tag{5}$$

$$\mathcal{U}(x,T) = g(x), \quad x \in (x_1, x_2), \tag{6}$$

$$\mathcal{U}(x,t) = 0, \quad x \in R \backslash (x_1, x_2), \quad t < T. \tag{7}$$

Here the pseudodifferential operator $L_x^{\mathbf{Q}}$ (acting on the variable x) is given by the formula

$$(L_x^{\mathbf{Q}} f)(x) = (\mathcal{F}^{-1}(-\psi^{\mathbf{Q}}(\cdot))\mathcal{F} f)(x), \tag{8}$$

where $\psi^{\mathbf{Q}}(\xi)$ is the characteristic exponent of the Lévy process X_t under the EMM $\mathbf{Q}$ and the Fourier transform is given (for $f \in L_1(\mathbb{R})$) as follows,

$$(\mathcal{F} f)(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x) dx. \tag{9}$$

Without loss of generality suppose that $x_1 = 0$ and $x_2 = a > 0$. Making the change of variable $\tau = T - t$, $u(x,\tau) = \mathcal{U}(x,t)$ we obtain the following problem,

$$\frac{\partial u(x,\tau)}{\partial \tau} + (r - L_x^{\mathbf{Q}})u(x,\tau) = 0, \ (x,\tau) \in (0,a) \times (0,\infty), \tag{10}$$

$$u(x,0) = g(x), \quad x \in (0,a), \tag{11}$$

$$u(x,\tau) = 0, \quad x \in R\backslash(0,a), \quad \tau \in (0,\infty). \tag{12}$$

Equation (2.10) is understood in the sense of generalized functions:

$$(u, (-\frac{\partial}{\partial \tau} + r - \widetilde{L}_x^{\mathbf{Q}})w) = 0 \tag{13}$$

for all $w \in S(R \times R)$ such that supp $w \subseteq (0,a) \times (0,\infty)$, where $S(R \times R)$ is the space of infinitely differentiable functions vanishing at infinity faster any negative power of $(x^2 + t^2)^{1/2}$ together with all derivatives. Here $u \in S'(R \times R)$ the set of all continuous linear functionals (distibutions) on $S(R \times R)$ and $\widetilde{L}_x^{\mathbf{Q}} := F^{-1}(-\psi^{\mathbf{Q}}(-\xi))F$. (For details see [15]).

3. Convolution equation and classes of symbols

Introduce the Laplace transform (LT) by variable τ and denote

$$v(x,w) := (\mathcal{L}u)(x,w) = \int_0^\infty e^{-w\tau} u(x,\tau) d\tau. \tag{1}$$

Applying integration by parts we obtain

$$\begin{aligned} \left(\mathcal{L}\frac{\partial u}{\partial t}\right)(x,w) &= u(x,\tau)e^{-w\tau}\Big|_0^\infty + w\int_0^\infty e^{-w\tau}u(x,\tau)d\tau \\ &= -u(x,0) + wv(x,w) = -g(x) + wv(x,w). \end{aligned}$$

Thus we pass from problem (2.10)–(2.12) to the following problem

$$(-L_x^{\mathbf{Q}} + r + w)v(x,w) = g(x), \quad x \in (0,a), \tag{2}$$

$$v(x,w) = 0 \quad x \in R\backslash(0,a). \tag{3}$$

We interpret the problem (2)–(3.3) as an operator equation considered in some H^s-spaces.

We introduce the corresponding notation. For $s \in R$ denote by $H^s(R)$ the space of distributions $f(\in S'(R))$ with finite norm defined by

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}} |(\mathcal{F}f)(\xi)|^2 (1 + |\xi|^2)^s d\xi.$$

Let U be an open subset of $\mathbb{R}$. Then denote by $H^s(U)$ the subspace of $H^s(R)$ consisting of distributions with supp $f \in \overline{U}$.

Introduce the set $C_0^\infty(U)$ of all functions f having all derivatives, with supp $f \in U$. It is well known that the closure $C_0^\infty(U)$ by norm of $H^s(R)$ coincides with $H^s(U)$. Suppose that $v(\cdot,w) \in H^{s_1}(0,a)$ and $g \in H^{s_2}(0,a)$. For such a function $v(\cdot,w)$ the condition (3.3) holds automatically.

Thus we can rewrite the problem (2)–(3.3) as the following equation,

$$P_{(0,a)}(\mathcal{F}^{-1}(\psi^{\mathbf{Q}}(\xi)+r+w)\mathcal{F})v(x,w)=g(x) \tag{4}$$

where $P_{(0,a)}$ is the operator of restriction to the interval $(0,a)$, $v(\cdot,w)\in H^{s_1}(0,a)$ and $g\in H^{s_2}(0,a)$. It should be noted that (3.4) is the convolution equation on the finite interval $(0,a)$ with the symbol $a(\xi,w):=\psi^{\mathbf{Q}}(\xi)+r+w$. This equation is understood in the sense of generalized functions analogously to (2.13).

Now we consider in more detail the properties of the function $\psi^{\mathbf{Q}}(\xi)$. Since X_t is a Lévy process under the measure $\mathbf{Q}$ then according to Lévy-Khintchine formula (2) we have

$$\psi^{\mathbf{Q}}(\xi)=\frac{1}{2}\sigma^2\xi^2-i\mu\xi+\varphi(\xi) \tag{5}$$

where $\sigma\geq 0$, $\mu\in R$, and

$$\varphi(\xi)=-\int_{-\infty}^{\infty}(e^{iu\xi}-1-i\xi uI_{(-1,1)}(u))\Pi^{\mathbf{Q}}(du) \tag{6}$$

with the measure $\Pi^{\mathbf{Q}}$ satisfying the condition

$$\left|\int_{-\infty}^{\infty}\frac{u^2}{1+u^2}\Pi^{\mathbf{Q}}(du)\right|<\infty. \tag{7}$$

Lemma 3.1. *(See* [15] *for example) For arbitrary* $\xi\in R,\quad \operatorname{Re}\psi^{\mathbf{Q}}(\xi)\geq 0.$

Proof. Since $\mathbf{Q}$ is a probability measure and $|e^{i\xi X_t}|=1$ for $\xi\in R$, we have from (2) that

$$|E_{\mathbf{Q}}[e^{iX_1}]|=|e^{-\psi^{\mathbf{Q}}(\xi)}|\leq 1.$$

That is,

$$e^{-\operatorname{Re}\psi^{\mathbf{Q}}(\xi)}\leq 1$$

and $\operatorname{Re}\psi^{\mathbf{Q}}(\xi)\geq 0$. ■

Lemma 3.2. *The characteristic function* $\psi^{\mathbf{Q}}(\xi)$ *is continuous for arbitrary* $\xi\in R$*, and the function* $\varphi(\xi)$ *has at infinity the following asymptotic property:*

$$\lim_{\xi\to\infty}\frac{\varphi(\xi)}{\xi^2}=0. \tag{8}$$

Proof. It is easy to see that the continuity of $\psi^{\mathbf{Q}}(\xi)$ follows from the representation (5)–(7).

Let $|\xi|$ be large. Then we have

$$|e^{iu\xi}-1-i\xi u|\leq\begin{cases}\text{const}\cdot|\xi u|^2, & \text{if} \quad |u|\leq|\xi|^{-1},\\ \text{const}\cdot|\xi u|, & \text{if} \quad |\xi|^{-1}\leq|u|\leq 1.\end{cases}$$

Then the following inequalities hold,

$$\begin{aligned}
|\varphi(\xi)| &\le \int_{|u|\ge 1} |e^{iu\xi}-1|\Pi^{\mathbf{Q}}(du) + \text{const}\,|\xi| \int_{|\xi|^{-1/2}<u<1} |u|\Pi^{\mathbf{Q}}(du) \\
&\quad + \text{const}\cdot\xi^2 \int_{|u|\le|\xi|^{-1/2}} u^2\Pi^{\mathbf{Q}}(du) \\
&\le 2\int_{|u|\ge 1} \Pi^{\mathbf{Q}}(du) + \text{const}\cdot|\xi|^{3/2} \int_{|\xi|^{-1/2}<|u|<1} u^2\Pi^{\mathbf{Q}}(du) \\
&\quad + \text{const}\cdot\xi^2 \int_{|u|\le|\xi|^{-1/2}} u^2\Pi^{\mathbf{Q}}(du).
\end{aligned}$$

According to (7), if $|\xi|\to\infty$ then

$$\begin{aligned}
&\int_{|u|\ge 1} \Pi^{\mathbb{Q}}(du) \quad \text{is bounded,} \\
&\int_{|\xi|^{-1/2}<u<1} u^2\Pi^{\mathbf{Q}}(du) \quad \text{is bounded, and} \\
&\int_{|u|<|\xi|^{-1/2}} u^2\Pi^{\mathbf{Q}}(du) \quad \text{tends to zero.}
\end{aligned}$$

Thus we obtain (8). ■

We use the following restrictions on the function $\psi^{\mathbf{Q}}(\xi)$ which naturally follow from Lemmas 3.1 and 3.2. Namely, the function $\psi^{\mathbf{Q}}(\xi)$ must have the form (5)

$$\psi^{\mathbf{Q}}(\xi) = \frac{1}{2}\sigma^2\xi^2 - i\mu\xi + \varphi(\xi) \tag{9}$$

where $\sigma\ge 0$, $\mu\in R$, and $\varphi(\xi)$ is a continuous function of $\xi\in R$ and there exists a number $\nu\in(0,2)$ such that the function $\varphi(\xi)$ has the following behavior at infinity,

$$\varphi(\xi) \sim |\xi|^{\nu}. \tag{10}$$

(The notation $\theta(\xi)\sim\eta(\xi)$ means that the quotients $\dfrac{|\theta(\xi)|}{|\eta(\xi)|}$ and $\dfrac{|\eta(\xi)|}{|\theta(\xi)|}$ are bounded by some constant for all $|\xi|$ large enough).

Thus we have the following cases of asymptotic behavior of the function $\psi^{\mathbf{Q}}(\xi)$ at infinity,

$$\begin{aligned}
\sigma &> 0, \quad \psi^{\mathbf{Q}}(\xi) \sim \frac{\sigma^2\xi^2}{2}, && (11)\\
\sigma &= 0, \quad 1\le\nu<2, \quad \psi^{\mathbf{Q}}(\xi)\sim|\xi|^{\nu}, && (12)\\
\sigma &= 0, \quad \mu=0, \quad 0<\nu\le 1, \quad \psi^{\mathbf{Q}}(\xi)\sim|\xi|^{\nu}, && (13)\\
\sigma &= 0, \quad \mu\ne 0, \quad 0<\nu<1, \quad \psi^{\mathbf{Q}}(\xi)\sim\xi. && (14)
\end{aligned}$$

This work is devoted to the cases (12)–(13). The cases (11), (14) and some others will be considered elsewhere.

Now we consider some examples of function $\psi^{\mathbf{Q}}(\xi)$ (we take the examples 3.1–3.5 from the book [15, chapter 3]).

Example 3.1 (Kobol Family). For Lévy processes from this family the characteristic exponent $\psi(\xi)$ can have the following forms,

i)

$$\begin{aligned}\psi(\xi) &= -i\mu\xi + c_+\Gamma(-\nu)[\lambda_-^\nu - (\lambda_- - i\xi)^\nu] \\ &\quad + c_-\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu]\end{aligned} \tag{15}$$

where $\nu \in (0,1)\cup(1,2)$, $\mu \in R$, $c_\pm > 0$, $\lambda_\pm > 0$, and $\Gamma(u)$ is the Euler Gamma-function;

ii)

$$\begin{aligned}\psi(\xi) &= -i\mu\xi + c_+[\ln(\lambda_- - i\xi) - \ln\lambda_-] \\ &\quad + c_-[\ln(\lambda_+ + i\xi) - \ln\lambda_+]\end{aligned} \tag{16}$$

where $\mu \in R$, $c_\pm > 0$, $\lambda_\pm > 0$;

iii)

$$\begin{aligned}\psi(\xi) &= -i\mu\xi + c_+[\lambda_- \ln(\lambda_-) - (\lambda_- - i\xi)\ln(\lambda_- - i\xi)] \\ &\quad + c_-[\lambda_+ \ln(\lambda_+) - (\lambda_+ + i\xi)\ln(\lambda_+ + i\xi)]\end{aligned} \tag{17}$$

where $\mu \in R$, $c_\pm > 0$, $\lambda_\pm > 0$.

Example i) corresponds to cases (12)–(14). The cases ii) and iii) have the following (non-power) behavior at infinity,

$$\psi(\xi) + \mu\xi \sim \ln\xi \tag{18}$$

and

$$\psi(\xi) \sim \xi\ln\xi. \tag{19}$$

Example 3.2 (Normal Tempered Stable Lévy Processes). In this case the characteristic exponent is

$$\psi(\xi) = -i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{\nu/2} - (\alpha^2 - \beta^2)^{\nu/2}] \tag{20}$$

where $\nu \in (0,2)$, $\mu \in R$, $\delta > 0$, $\beta \in R$, $\alpha > |\beta|$ (see (3.13)–(14)).

Example 3.3 (Normal Inverse Gaussian Processes). If we put in (20) $\nu = 1$ we obtain the characteristic exponent of a normal inverse Gaussian Process

$$\psi(\xi) = -i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{1/2} - (\alpha^2 - \beta^2)^{1/2}] \tag{21}$$

(see (3.13)–(3.14)).

Example 3.4 (Variance Gamma Processes). The characteristic exponent for this case is

$$\psi(\xi) = -i\mu\xi + c[\ln(\alpha^2 - (\beta + i\xi)^2) - \ln(\alpha^2 - \beta^2)] \tag{22}$$

where $c > 0$, $\beta \in R$, $\alpha > |\beta| > 0$ (see (18)).

Example 3.5 (Generalized Hyperbolic Processes). For this case the characteristic function is

$$\exp(\psi(\xi)) = e^{i\mu\xi}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+i\xi)^2}\right)^{\lambda/2}\frac{K_\lambda(\delta\sqrt{\alpha^2-(\beta+i\xi)^2})}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})} \tag{23}$$

where $\mu \in R$, $\beta \in R$, $\alpha > |\beta| > 0$, $\delta > 0$, $\lambda \in R$, and $K_\lambda(u)$ is the modified Bessel function of the third kind with index λ. An integral representation of $K_\lambda(u)$ is given by

$$K_\lambda(u) = \frac{1}{2}\int_0^\infty y^{\lambda-1}\exp[-0.5u(y+y^{-1})]dy.$$

Example 3.6 (Poisson Processes). For a Poisson processes we have the following characteristic exponent (see, for example, [28])

$$\psi(\xi) = c(1-e^{i\xi}). \tag{24}$$

It is easy to see that the Poisson process of the kind (3.24) has characteristic triplet $(0, c, c\Pi_1)$ where Π_1 is a discrete measure which is concentrated in the one point $u_0 = 1$ with weight equal to 1.

Let Π_d be discrete measure which is concentrated in the points $\{\mu_1, \mu_2, \dots, \mu_n\}$ with weights $\{d_1, d_2, \dots, d_n\}$ respectively. Suppose that the Lévy process X_t has the characteristic triplet (σ, μ, Π_d). According to formula (2) the characteristic exponent of X_t has the following form,

$$\psi(\xi) = \frac{1}{2}\sigma^2\xi^2 - i\left(\mu - \sum_{|u_j|<1} d_j u_j\right)\xi - \sum_{j=1}^{n}(e^{iu_j\xi}-1)d_j. \tag{25}$$

It should be noted that (ACP)-condition does not hold in this case. In spite of that we can use the system (2.5)-(2.7) for a finding of option price (see remark 2.1 d) of [15], p. 64).

Example 3.7 (Rational Characteristic Exponent). Let the measure Π_{r_1} be given by the following formula,

$$\Pi_{r_1}(dx) = \lambda_+c_+e^{\lambda_+x}\chi_{(-\infty,0)}(x)dx + \lambda_-c_-e^{-\lambda_-x}\chi_{(0,\infty)}dx \tag{26}$$

where $c_\pm > 0$, $\lambda_\pm > 0$, and $\chi_{(-\infty,0)}(x)$, $\chi_{(0,\infty)}(x)$ are characteristic functions of the semi-axes.

Consider the Lévy process X_t with the triplet (σ, μ, Π_{r_1}). The characteristic exponent in this case is

$$\psi_{r_1}(\xi) = \frac{\sigma^2}{2}\xi^2 - i\gamma'\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} - \frac{ic_-\xi}{\lambda_- - i\xi} \tag{27}$$

where

$$\gamma' = \gamma - \left(c_+\lambda_+\int_{-1}^{0} ue^{\lambda_+ u}du + c_-\lambda_-\int_{0}^{1} ue^{-\lambda_- u}du\right)$$
$$= \gamma - c_+\frac{(1+\lambda_+)e^{-\lambda_+}-1}{\lambda_+} + c_-\frac{(1+\lambda_-)e^{-\lambda_-}-1}{\lambda_-}.$$

It should be noted that replacement of the factors $e^{\pm\lambda_\pm x}$ by factors of the form $\sum_{j=1}^{m} d_j^{\pm}e^{\pm\lambda_{j,\pm}x}$ in (26) provides more general rational characteristic exponent.

4. Reducing to Modified Wiener-Hopf Equation. Necessary Information from Toeplitz Operator theory

4.1.

Modified Wiener Hopf Equation

Our basic equation (3.4) is defined on the interval $(0, a)$. Extend it to the whole real axis R, that is, rewrite this equation in the form

$$v_0(x,w) + F^{-1}(\psi^{\mathbf{Q}}(\xi) + r + w)Fv(x,w) = g(x) \tag{1}$$

where $v_0(x,w) \in H^{s_2}((a,\infty)\cup(-\infty,0))$.

If $s_2 \in (-1/2, 1/2)$, then (see [38], [40])

$$H^{s_2}((a,\infty)\cup(-\infty,0)) = H^{s_2}(a,\infty)\oplus H^{s_2}(-\infty,0) \tag{2}$$

Thus in this case equation (4.1) is

$$v_1(x,w) + F^{-1}(\psi^{\mathbf{Q}}(\xi) + r + w)Fv(x,w) + v_2(x,w) = g(x) \tag{3}$$

where

$$g(x) \in H^{s_2}(0,a) \tag{4}$$
$$v_1(x,w) \in H^{s_2}(a,\infty) \tag{5}$$
$$v_2(x,w) \in H^{s_2}(-\infty,0) \tag{6}$$

If $s_2 \notin (-1/2, 1/2)$, the decomposition (4.2) does not hold. However, in this case we suppose that case $v_0 = v_1 + v_2$ where v_1, v_2 satisfy (4.5), (4.6) like we have in the good case. This additional requirement holds if the function $g(x)$ is good enough, for example (as we will see below) when $g(x)$ satisfies (4.4)

Apply the Fourier transform to equation (3). Denote

$$\Phi_a^-(\xi,w) := (\mathcal{F}v)(\xi,w) \qquad (\in L_2(\mathbb{R},s_1); \tag{7}$$
$$e^{-ia\xi}\Phi^-(\xi,w) := (\mathcal{F}v_1)(\xi,w) \qquad (\in L_2(\mathbb{R},s_2); \tag{8}$$
$$\Phi^+(\xi,w) := (\mathcal{F}v_2)(\xi,w) \qquad (\in L_2(\mathbb{R},s_2)); \tag{9}$$
$$\hat{g}(\xi) = (\mathcal{F}g)(\xi) \qquad (\in L_2(\mathbb{R},s_2); \tag{10}$$

where $L_2(\mathbb{R}, s)$ is Hilbert space with the norm

$$\|\Phi\|_{L_2^s} = \int_{-\infty}^{\infty} |\Phi(\xi)|^2 (1+\xi^2)^s d\xi.$$

(Note that the "+" sign in the notation $\Phi^+(\xi, w)$ means that this function is analytic in the upper half-plane. Analogously the "-" sign in $\Phi_a^-(\xi, w)$ and $\Phi^-(\xi, w)$ means that these functions are analytic in the lower half-plane.)

Thus we obtain the following boundary value problem

$$e^{-ia\xi}\Phi^-(\xi, w) + a(\xi, w)\Phi_a^-(\xi, w) + \Phi^+(\xi, w) = \hat{g}(\xi). \tag{11}$$

This problem is called ([39]) the modified Wiener-Hopf equation and its solution is a triple $(\Phi^-, \Phi_a^-, \Phi^+)$ of unknown functions. We emphasize that these unknown functions are not arbitrary functions from $L_2(\mathbb{R}, s_{1,2})$. Namely $\Phi_a^-(\xi, w)$ is the Fourier Transform of a function with support belonging to $(0, a)$, and $\Phi^{\mp}(\xi, w)$ are Fourier Transforms of functions with support in the semi-axes $(0, \infty)$ and $(-\infty, 0)$ respectively. The classes where the solution of (10) are looked for will be introduced in the Section 5.

4.2.

Toeplitz Operators

We need some results from Toeplitz operator theory. Introduce the so-called analytic projectors

$$P^+ := \mathcal{F}^{-1}\chi_{(0,\infty)}\mathcal{F} \quad \text{and} \quad P^- := \mathcal{F}^{-1}\chi_{(-\infty,0)}\mathcal{F}.$$

The projectors $P^{\pm}$ are bounded linear operators in the spaces $L_2(\mathbb{R}, s)$ for $s \in (-1/2, 1/2)$ ([40], [41]).

Denote

$$L_2^{\pm}(\mathbb{R}, s) = P^{\pm}(L_2(\mathbb{R}, s)).$$

It is easy to see (see also [40], [41]) that

$$P^{\pm 2} = P^{\pm}, \quad P^+P^- = P^-P^+ = 0, \quad P^+ + P^- = I$$

where 0 and I are the zero and identity operators.

Let further $L_\infty(\mathbb{R})$ be the space of all measurable essentially bounded functions on the real axis $\mathbb{R}$ with the norm

$$\|a\|_{L_\infty(\mathbb{R})} = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |a(x)| < \infty.$$

The operator

$$T(a) := P^+aP^+ : L_2^+(\mathbb{R}, s) \to L_2^+(\mathbb{R}, s)$$

is called a Toeplitz operator with symbol $a(x)$. If $a \in L_\infty(\mathbb{R})$ then $T(a)$ is a bounded operator on $L_2^+(\mathbb{R}, s)$ (for $s \in (-1/2, 1/2)$) and the conjugate operator $T^*(a) = T(\overline{a})$ also is bounded on the same spaces.

Definition 4.1. *The operator A acting on Hilbert space is called normally solvable if the subspace* $\operatorname{im} A$ *is closed, i.e.,* $\operatorname{im} A = \overline{\operatorname{im} A}$.

We will use the following well-known fact from functional analysis.

Lemma 4.1. *If the operator A is normally solvable, then the Hilbert space H may be represented as the following direct sum,*

$$\operatorname{im} A \oplus \ker A^* = H.$$

Definition 4.2. *An operator A acting in Hilbert space H is called left-(right)- invertible if there exists an operator A_l^{-1} (A_r^{-1}) bounded on H such that $A_l^{-1}A = I$ ($AA_r^{-1} = I$). A is called an invertible operator if there exists an operator A^{-1} bounded on H such that $AA^{-1} = A^{-1}A = I$.*

It should be noted that a one-side invertible operator is normally solvable. Moreover if A is left-invertible, then $\ker A = \{0\}$; if A is right-invertible, then $\operatorname{im} A = H$.

Introduce the following well-known subspace of $L_\infty(\mathbb{R})$ in the theory of Toeplitz operators. Namely, $H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}})$ is the set of all essentially bounded functions $f(x)$ representable in the form $f(x) = h(x) + c(x)$ where $h(x)$ is bounded and analytic in the upper half-plane. That is, $h(x)$ belongs to the Hardy space $H^\infty(\mathbb{R})$, and $c(x)$ belongs to the space $C(\dot{\mathbb{R}})$, the set of all continuous on $\mathbb{R}$ functions such that $\lim\limits_{x\to+\infty} c(x) = \lim\limits_{x\to-\infty} c(x) = c_0 \in \mathbb{C}$.

We formulate the following well-known facts from the theory of Toeplitz operators connected with class $H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}})$ ([40], [41]).

Lemma 4.2. *Let the symbol $a(x) \in H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}})$ ($\in \overline{H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}})}$) and $\operatorname{ess\,inf}\limits_{x\in\mathbb{R}} |a(x)| > 0$. Then the operator $T(a)$ is left-invertible (right-invertible) in the space $L_2^+(\mathbb{R}, s)$, for $|s| < 1/2$.*

Lemma 4.3 (Sarason Lemma ([42])). *Let the function $c(x)$ be continuous on $\mathbb{R}$ and let there exist $\lim\limits_{x\to+\infty} c(x) = c_+$ and $\lim\limits_{x\to-\infty} c(x) = c_-$ (in general $c_+ \neq c_-$), and let $\lambda > 0$ ($\lambda < 0$). Then we have*

$$e^{i\lambda x}c(x) \in H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}}) \quad (\in \overline{H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}})}).$$

4.3.
Sectoriality

Our work is based essentially on the concept of sectoriality. Lemma 3.1 makes this possible. This subsection is devoted to necessary information from the theory of sectorial operators.

Definition 4.3. *A linear bounded operator acting on a Hilbert space H is called a sectorial operator if*

$$\inf_{\|x\|_H=1} (Ax, x) := \varepsilon > 0 \tag{12}$$

where (Ax, y) denotes the scalar product in H.

If $a(x) \in L_\infty(\mathbb{R})$ then the operator of multiplication by the function $a(x)$ in the space $L_2(\mathbb{R}, s)$ is sectorial if and only if

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}} \operatorname{Re} a(x) = \varepsilon > 0. \tag{13}$$

Definition 4.4. *We will call a function $a(x) \in L_\infty(\mathbb{R})$ sectorial if exists a number $\theta \in (-\pi, \pi)$ such that for the function $a_\theta(x) := e^{i\theta}a(x)$ the condition (11) holds.*

We formulate the famous result of Brown and Halmos (see, for example, [25, Theorem 2.2]).

Theorem 4.1. *Let A be a sectorial operator on a Hilbert space H. Then the operator A is invertible and the following estimate holds for the norm of the inverse operator,*

$$\|A^{-1}\|_H \le 2\varepsilon^{-1}$$

where ε is the value from (4.12).

Let now G be a subspace of the Hilbert space $L_2^+(\mathbb{R}, s)$ and let $\mathcal{P}_G$ be the orthoprojector onto the space G. This means that an arbitrary function $f(x) \in L_2^+(\mathbb{R}, s)$ can be represented uniquely in the form

$$f(x) = g_1(x) + g_2(x) \tag{14}$$

where $g_1(x) \in G$, $g_2(x) \in G^\perp$, and $G^\perp$ denotes the orthogonal complement of the space G in $L_2^+(\mathbb{R}, s)$. Thus the following equation holds,

$$P^+ = \mathcal{P}_G^\perp + \mathcal{P}_G$$

where $\mathcal{P}_G^\perp$ is the orthoprojector onto $G^\perp$. Consider the operator

$$D = \mathcal{P}_G^\perp + P^+ a\mathcal{P}_G : L_2^+(\mathbb{R}, s) \to L_2^+(\mathbb{R}, s) \tag{15}$$

where the function a belongs to $L_\infty(\mathbb{R})$.

Theorem 4.2. *Let function $a(\in L_\infty(\mathbb{R}))$ be sectorial. Then the operator D (15) is invertible and for the solution x of the equation*

$$Dx = f, \quad f \in L_2^+(\mathbb{R}, s), \tag{16}$$

there holds the following estimate,

$$\|x_1\|_{L_2(\mathbb{R},s)} \le 2\varepsilon^{-1}\|f_1\|_{L_2(\mathbb{R},s)} \tag{17}$$

where $x_1 = \mathcal{P}_G x$, $f_1 = \mathcal{P}_G f$, and ε is the value from (11).

Proof. Consider the operator

$$D_1 := \mathcal{P}_G a\mathcal{P}_G : G \to G.$$

and

$$D_{1,\theta} = P_G a_\theta P_G : G \to G$$

where the function $a_\theta(x)(= e^{i\theta}a(x))$ and the number θ are from definition 4.4.

We show that $D_{1,\theta}$ operator is sectorial. Let $x_1 \in G$ then

$$(\mathcal{P}_G a_\theta x_1, x_1)_{L_2(\mathbb{R})} = (a_\theta x_1, x_1)_{L_2(\mathbb{R})}.$$

Then

$$\begin{aligned}\operatorname{Re}(\mathcal{P}_G a_\theta x_1, x_1) &= \operatorname{Re}\left(\int_{-\infty}^{\infty} a_\theta(t)|x_1(t)|^2 dt\right) \\ &= \int_{-\infty}^{\infty} (\operatorname{Re} a_\theta(t))|x_1(t)|^2 dt \geq \varepsilon(x_1, x_1).\end{aligned}$$

Thus the operator $D_{1,\theta}$ is invertible and according to Theorem 4.1 we have

$$\|D_{1,\theta}^{-1}\|_{L_2(\mathbb{R},s)} \leq 2\varepsilon^{-1}$$

where ε is the value from (11).

Since $D_{1,\theta} = e^{i\theta} D_1$, then

$$||D_1^{-1}||_{L_2(\mathbb{R},s)} < 2\varepsilon^{-1} \tag{18}$$

Now rewrite the equation (16) in the form

$$x_2 + P^+ a x_1 = f_1 + f_2 \tag{19}$$

where $x_2 = \mathcal{P}_G^\perp x$ and $f_2 = \mathcal{P}_G^\perp f$. Applying the projector $\mathcal{P}_G$ to the last equality we get

$$D_1 x_1 = f_1. \tag{20}$$

Since D_1 is invertible, the equation (20) has a unique solution

$$x_1 = D_1^{-1} f_1$$

and according to (18) we have (17). Further applying the projector $\mathcal{P}^\perp$ to (19) we get

$$x_2 + \mathcal{P}_G^\perp a x_1 = f_2.$$

Therefore $x_2 = f_2 - \mathcal{P}_G^\perp a D_1^{-1} f_1$. Thus for arbitrary $f \in L_2^+(\mathbb{R})$ the equation (16) has a unique solution in the form

$$x = (D_1^{-1}\mathcal{P}_G + \mathcal{P}_G^\perp - \mathcal{P}_G^\perp a D_1^{-1}\mathcal{P}_G) f$$

and consequently the operator D is invertible. ■

5. Unique Solvability of Modified Wiener-Hopf Equation in space $L_2(\mathbb{R}, s)$

This section is central in this work. In order to obtain the theorem of solvability we apply the Matrix Riemann Boundary Problem approach (originally worked out in [24]–[27]) for some diffraction problems. It should be noted that this approach suits perfectly for barrier option problems.

Assume that function $\psi^{\mathbf{Q}}(\xi)$ satisfies the following conditions (see (9), (10))

$$\psi^{\mathbf{Q}}(\xi) = \frac{1}{2}\sigma^2\xi^2 - i\mu\xi + \varphi(\xi). \tag{1}$$

We suppose (see (12), (13)) that

$$\sigma = 0, \tag{2}$$

that there exists such $\nu \in (0,2)$ that the function

$$c(\xi) := \frac{\varphi(\xi)}{(1+\xi^2)^{\nu/2}} \in L_\infty(\mathbb{R}), \tag{3}$$

for some $M > 0$ satisfies

$$\inf_{|\xi|\geq M} \operatorname{Re} c(\xi) = \varepsilon_1 > 0, \tag{4}$$

and that

$$\begin{cases} \text{if} \quad \mu \neq 0 \quad \text{then} \quad 1 < \nu < 2, \\ \text{if} \quad \mu = 0 \quad \text{then} \quad 0 < \nu < 2. \end{cases} \tag{5}$$

Finally we assume that

$$r + \operatorname{Re} w \geq \varepsilon_2 > 0. \tag{6}$$

It should be noted that the interest rate of the bond r is positive and the complex number w lies on the contour L of the inverse Laplace transform. Very often $L = \{z \in \mathbb{C} : \operatorname{Re} z = -\delta\}$ where $\delta (\geq 0)$ is as small as we wish. Thus the condition (6) is natural.

Introduce the function

$$c(\xi, w) := \frac{(\psi^{\mathbf{Q}}(\xi) + r + w)}{(1+\xi^2)^{\nu/2}}. \tag{7}$$

Lemma 5.1. *Let the conditions (1)-(6) hold. Then the function $c(\cdot, \xi)$ is sectorial, and if the value ε_2 in (6) is independent of w then there exists a number ε independent of w such that*

$$\inf_{\xi\in\mathbb{R}} \operatorname{Re} c(\xi, w) \geq \varepsilon > 0. \tag{8}$$

Proof. According to Lemma 3.1 and conditions (2), (6) we have for $\xi \in \mathbb{R}$

$$\operatorname{Re} c(\xi, w) = \operatorname{Re}\left(\frac{\varphi(\xi)}{(1+\xi^2)^{\nu/2}}\right) + \frac{r + \operatorname{Re} w}{(1+\xi^2)^{\nu/2}} > 0.$$

According to (6),

$$\inf_{|\xi|\leq M} \operatorname{Re} c(\xi, w) \geq \frac{\varepsilon_2}{(1+M^2)^{\nu/2}}.$$

Further according to (4) we get

$$\inf_{|\xi|\geq M} \operatorname{Re} c(\xi, w) \geq \varepsilon_1.$$

Now set

$$\varepsilon = \min\left(\frac{\varepsilon_2}{(1+M^2)^{\nu/2}}, \varepsilon_1\right). \tag{9}$$

Then we obtain (8).

Finally with the help of (3) and (5) we see that $c(\xi, w) \in L_\infty(\mathbb{R})$. Thus $c(\xi, w)$ is sectorial. ■

We see that according to Lemma 5.1 if the condition (5.6) holds then function $c(\xi, w)$ is sectorial with $\theta = 0$. It should be noted that $c(\xi, w)$ could be sectorial even when the condition (5.6) does not hold. In particular we need the following result.

Lemma 5.2. *Let the conditions (5.1)-(5.5) hold and for $w(\neq 0)$ suppose that*

$$|\arg w| \leq \frac{\pi}{2} + \theta_0, \quad \theta_0 > 0, \tag{10}$$

Then there exists a number θ_0 (small enough) such that the function $c(w)$ is sectorial with the same ε (see definition 4.4) for all w satisfying the condition (5.10).

Proof. According to Lemma 5.1 we have that the statement is true for the region $|\arg w| \leq \frac{\pi}{2}$.

The function $c(\xi, 0)$ is sectorial with parameter $\theta = 0$. This means that there exists a number $\theta_0 > 0$ such that the set $J_0 := \{z \in |z = c(\xi, 0), \xi \in R\}$ lies strictly within the region $|\arg z| < \frac{\pi}{2} - \theta_0$.

Consider the case that

$$\frac{\pi}{2} < \arg w \leq \frac{\pi}{2} + \theta_0$$

Let $\varepsilon_1(> 0)$ be the distance between J_0 and the line

$$R_{-\frac{\pi}{2}+\theta_0} = \left\{ z \in \mathbb{C} | z = re^{i(-\frac{\pi}{2}+\theta_0)}, r \in \mathbb{R} \right\}$$

Then the distance between the set

$$J_w := \{z \in R | z = c(\xi, w), \xi \in R\}$$

and the semiplane

$$-\frac{\pi}{2} + \varphi_0 \leq \arg z \leq \frac{\pi}{2} + \varphi_0$$

is no smaller then ε_1, since

$$c(\xi, w) = c(\xi, 0) + \frac{|w| e^{i \arg w}}{(1 + \xi^2)^{v/2}}$$

Thus the function $c(\xi, w)$ is sectorial with parameters $\theta = -\theta_0$ and $\varepsilon = \varepsilon_1$.

The case $-\frac{\pi}{2} - \varphi_0 \leq \arg w < -\frac{\pi}{2}$ is considered analogously.

It should be noted that the hypothesis of Lemmas 5.1-5.2 hold for Lévy processes of the Kobol family in the case (15), for normal tempered stable Lévy processes (20) and for normal inverse Gaussian processes (21).

Now consider equation (10). It is convenient for us to make a change of variable ($\xi \mapsto -\xi$) and denote

$$\widetilde{\Phi}^{\pm}(\xi, w) = \Phi^{\mp}(-\xi, w), \quad \widetilde{\Phi}^{+}_{a}(\xi, w) = \Phi^{-}_{a}(-\xi, w).$$

Then we can rewrite (10) in the form

$$e^{ia\xi} \widetilde{\Phi}^{+}(\xi, w) + (1 + \xi^2)^{\nu/2} \widetilde{c}(\xi, w) \widetilde{\Phi}^{+}_{a}(\xi, w) + \widetilde{\Phi}^{-}(\xi, w) = \hat{g}(-\xi), \tag{11}$$

where conditions (5.1)-(5.6) are satisfied and $\widetilde{c}(\xi, w) = c(-\xi, w)$ (see 5.7).

Furthermore we assume that conditions (6)–(9) hold for $s_1 = \nu/2 + s$, $s_2 = -\nu/2 + s$ where $|s| < \frac{1}{2}$. That is,

$$\widetilde{\Phi}^{\pm}(\xi, w) \in L^{\pm}_{2}(\mathbb{R}, -\nu/2 + s); \tag{12}$$

$$\widetilde{\Phi}_a^+(\xi,w) \in L_2^+(\mathbb{R},\nu/2+s). \tag{13}$$

Consider the so-called Wiener-Hopf factorization of the function $\gamma(\xi) := (1+\xi^2)^{\nu/2}$,

$$\gamma(\xi) = (1+i\xi)^{\nu/2}(1-i\xi)^{\nu/2} := \gamma_-(\xi)\gamma_+(\xi).$$

The cuts of the functions $\gamma_\pm(\xi) := (1 \mp i\xi)^{\nu/2}$ pass along the rays $\Gamma_\pm = \{z \in \mathbb{C} : z = \mp is, s \in [1,\infty)\}$ respectively. Thus the function $\gamma_+(\xi)$ is analytic in the upper half-plane and $\gamma_-(\xi)$ is analytic in the lower half-plane.

Divide all terms of (5.11) by $\gamma_-(\xi)$ and write

$$\Psi_a^+(\xi,w) \;:=\; \gamma_+(\xi)\widetilde{\Phi}_a^+(\xi,w); \tag{14}$$

$$\Psi^\pm(\xi,w) \;:=\; \frac{\widetilde{\Phi}^\pm(\xi,w)}{\gamma_\pm(\xi)}. \tag{15}$$

Then we obtain

$$e^{ia\xi}u(\xi)\Psi^+(\xi,w) + \widetilde{c}(\xi,w)\Psi_a^+(\xi,w) + \Psi^-(\xi,w) = \frac{\hat{g}(-\xi)}{\gamma_-(\xi)} \tag{16}$$

where

$$u(\xi) := \frac{\gamma_+(\xi)}{\gamma_-(\xi)} = \left(\frac{1-i\xi}{1+i\xi}\right)^{\nu/2}. \tag{17}$$

It is easy to see (see, for example, [40]) that

$$\Psi_a^+(\xi) \;\in\; L_2^+(\mathbb{R},s); \tag{18}$$

$$\Psi^\pm(\xi) \;\in\; L_2^\pm(\mathbb{R},s). \tag{19}$$

It should be noted that the functions $\widetilde{\Phi}_a^+(\xi,w)$ and $\Psi_a^+(\xi,w)$ belong to narrower classes of functions than $L_2^+(\mathbb{R},\nu/2+s)$ and $L_2^+(\mathbb{R},s)$ respectively. Namely, the following statement holds.

Lemma 5.3. i) *The class of functions where the unknown function $\widetilde{\Phi}_a^+(\xi,w)$ is looked for coincides with the set of "+"-components of the solutions of the boundary value problem*

$$e^{-ia\xi}\Phi_a^+(\xi) = \Phi_a^-(\xi), \quad a > 0, \tag{20}$$

where $\Phi_a^\pm(\xi) \in L_2^\pm(\mathbb{R},\nu/2+s)$.

ii) *The class of functions where the unknown function $\Psi_a^+(\xi,w)$ is looked for coincides with the set of "+"-components of solutions of the boundary value problem*

$$e^{-ia\xi}\overline{u}(\xi)\Psi_a^+(\xi) = \Psi_a^-(\xi), \quad a > 0, \tag{21}$$

where $\Psi_a^\pm(\xi) \in L_2^\pm(\mathbb{R},s), |s| < 1/2$.

Proof. The statement i) is well-known ([43]-[44]).

We pass to the proof of ii). Multiply both sides of (20) by $\gamma_-(\xi)$ and denote

$$\Psi_a^+(\xi) := \gamma_+(\xi)\Phi_a^+(\xi) \quad \text{and} \quad \Psi_a^-(\xi) = \gamma_-(\xi)\Phi_a^-(\xi).$$

Then we obtain (21). ∎

The problems (20) and (21) are called Riemann Boundary Value Problems with coefficients $e_a(\xi) := e^{-ia\xi}$ and $\overline{u_a}(\xi) := e^{-ia\xi}\overline{u}(\xi)$ respectively. Moreover the set of all functions $\Phi_a^+(\xi)$ $(\in L_2^+(\mathbb{R}, \nu/2+s))$ satisfying problem (20) coincides with the kernel of the Toeplitz operator T_{e_a} in the space $L_2^+(\mathbb{R}, \nu/2+s)$. Analogously the set of all functions $\Psi_a^+(\xi)$ satisfying the problem (21) coincides with the subspace $\ker T_{\overline{u}_a}\Big|_{L_2^+(\mathbb{R},s)}$.

Thus the components of the solution of the problem (11) are looked for in the following spaces

$$\widetilde{\Phi}_a^+(\xi, w) \quad \in \quad \ker T_{e_a}\Big|_{L_2^+(\mathbb{R},\nu/2+s)}; \tag{22}$$

$$\widetilde{\Phi}^{\pm}(\xi, w) \quad \in \quad L_2^{\pm}(\mathbb{R}, -\nu/2 + s). \tag{23}$$

Analogously the components of solution of the problem (16) are looked for in the spaces

$$\Psi_a^+(\xi, w) \quad \in \quad \ker T_{\overline{u}_a}\Big|_{L_2^+(\mathbb{R},s)}; \tag{24}$$

$$\Psi^{\pm}(\xi, w) \quad \in \quad L_2^{\pm}(\mathbb{R}, s). \tag{25}$$

Apply the projector P^+ to all terms of equation (16). Then we have

$$(T_{u_a}\Psi^+)(\xi, w) + P^+(\widetilde{c}(\xi, w)\Psi_a^+(\xi, w) = f^+(\xi) \tag{26}$$

where T_{u_a} is the Toeplitz operator with symbol

$$u_a(\xi) := e^{ia\xi}u(\xi), \tag{27}$$

and

$$f^+(\xi) = P^+(\hat{g}(-\xi)/\gamma_-(\xi)), \tag{28}$$

$$\Psi_a^+(\xi, w) \in \ker T_{\overline{u}_a}\Big|_{L_2^+(\mathbb{R},s)}, \tag{29}$$

$$\Psi^+(\xi, w) \in L_2^+(\mathbb{R}, s). \tag{30}$$

It is easy to observe that the problem (11), (22), (23) has a solution if and only if the problem (26)–(30) has a solution as well. Further the components of the solution of the first problem relate to the components of solution of the second problem by means of formulae (14)–(15).

Consider the function $u(\xi)$. It is easy to see that $u(\xi)$ is continuous on $\mathbb{R}$ and $\lim\limits_{\xi\to\pm\infty} u(\xi) = e^{\mp i\pi\nu/2}$. Thus according to Lemma 4.3 $u_a(\xi) \in H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}})$. Consequently according to Lemma 4.2 the Toeplitz operator T_{u_a} is left-invertible and according to Lemma 4.1 we have $\operatorname{im} T_{u_a} \oplus \ker T_{\overline{u}_a} = L_2^+(\mathbb{R}, s)$ since $T_{u_a}^* = T_{\overline{u}_a}$. Associate with this decomposition the pair of orthogonal projectors $\mathcal{P}_{u_a}^\perp$ and $\mathcal{P}_{u_a}$ ($\mathcal{P}_{u_a}(L_2^+(\mathbb{R}, s)) = \ker T_{\overline{u}_a}$, $\mathcal{P}_{u_a}^\perp(L_2^+(\mathbb{R}, s)) = \operatorname{im} T_{u_a}$) and consider the operator

$$D_{u_a} := \mathcal{P}_{u_a}^\perp + P^+\widetilde{c}(\xi, w)\mathcal{P}_{u_a} : L_2^+(\mathbb{R}, s) \to L_2^+(\mathbb{R}, s). \tag{31}$$

Associate with the operator (5.31) the following operator equation

$$(D_{u_a}Y^+)(\xi) = f^+(\xi), \quad Y^+(\xi) \in L_2^+(\mathbb{R}, s) \tag{32}$$

where $f^+(\xi)$ is defined by (28).

Lemma 5.4. *The problem (26)–(30) has a solution if and only if the equation (32) has a solution as well. Moreover if $Y^+(\xi)$ satisfies (32) then the following functions*

$$\Psi_a^+(\xi) = (\mathcal{P}_{u_a}Y^+)(\xi); \tag{33}$$
$$\Psi^+(\xi) = T_{u_a}^{-1}(\mathcal{P}_{u_a}^{\perp}Y^+)(\xi) \tag{34}$$

are a solution of 5.26.

Here $T_{u_a}^{-1}$ is a left inverse of operator T_{u_a}.

Proof. Let $Y^+(\xi)$ be a solution of (32). Taking into account that for function $f(\xi)$ belonging to $\operatorname{im} T_{u_a}$, $(T_{u_a}T_{u_a}^{-1}f)(\xi) = f(\xi)$ and substituting (33)–(34) to equation (26) we obtain

$$\begin{aligned}&T_{u_a}T_{u_a}^{-1}(\mathcal{P}_{u_a}^{\perp}Y^+)(\xi) + P^+(\widetilde{c}(\xi,w)(\mathcal{P}_{u_a}Y^+)(\xi))\\ &= (\mathcal{P}_{u_a}^{\perp}Y^+)(\xi) + P^+(\widetilde{c}(\xi,w)(\mathcal{P}_{u_a}Y^+)(\xi)) = (D_{u_a}Y^+)(\xi) = f^+(\xi).\end{aligned}$$

Conversely, let $(\Psi_a^+(\xi), \Psi^+(\xi))$ be a solution of (26). Then it is easy to check that the function

$$Y^+(\xi) := (T_{u_a}\Psi^+)(\xi,w) + \Psi_a^+(\xi)$$

is a solution of (32). ■

Theorem 5.1. *Let the function $c(\xi,w)$ (7) satisfy conditions (1)–(5) and w belong the region (5.10). Then the following statements are true:*

i) *The operator D_{u_a} (31) is invertible and for the solution of (32) the following estimate holds*

$$\|\mathcal{P}_{u_a}Y^+\|_{L_2(\mathbb{R},s)} \le 2\varepsilon^{-1}\|\mathcal{P}_{u_a}f^+\|_{L_2(\mathbb{R},s)}.$$

where ε does not dependent of w.

ii) *The problem (26)–(30) has the unique solution*

$$\Psi_a^+(\xi,w) = (\mathcal{P}_{u_a}D_{u_a}^{-1}f^+)(\xi), \quad \Psi^+(\xi,w) = (T_{u_a}^{-1}\mathcal{P}_{u_a}^{\perp}D_{u_a}^{-1}f^+)(\xi). \tag{35}$$

iii) *The problem (11), (22), (23) has the unique solution*

$$\widetilde{\Phi}_a^+(\xi,w) = \frac{1}{\gamma^+(\xi)}\left(\mathcal{P}_{u_a}D_{u_a}^{-1}P^+\frac{\hat{g}(-\xi)}{\gamma_-(\xi)}\right); \tag{36}$$
$$\widetilde{\Phi}^+(\xi,w) = \gamma^+(\xi)\left(T_{u_a}^{-1}\mathcal{P}_{u_a}^{\perp}D_{u_a}^{-1}P^+\frac{\hat{g}(-\xi)}{\gamma_-(\xi)}\right); \tag{37}$$
$$\widetilde{\Phi}^-(\xi,w) = \gamma_-(\xi)\Psi^-(\xi,w); \tag{38}$$

where the function $\Psi^-(\xi,w)$ can be found from the relation (16).

Proof. The statement i) follows directly from Theorem 4.2 and Lemma 5.2.

ii) This statement follows from i) and Lemma 5.4 since the function $Y^+(\xi) := (D_{u_a}^{-1}f^+)(\xi)$ is the unique solution of equation (32).

iii) It is easy to see that problems (11), (22), (23) and (26)–(30) have solutions simultaneously and they are connected according to formulae (14)–(15). Moreover the triple $(\Psi^+, \Psi_a^+, \Psi^-)$ satisfies the equation (16) if and only if the pair (Ψ^+, Ψ_a^+) satisfies the equation (26) and

$$\Psi^-(\xi, w) = \left(P^- \frac{\hat{g}(-\xi)}{\gamma_-(\xi)}\right) - P^-(u_a(\xi)\Psi^+(\xi, w)) - P^-(\widetilde{c}(\xi, w)\Psi_a^+(\xi, w)). \quad \blacksquare$$

6. Unique Solvability of the problem (2.10)-(2.12) and the price of Double Barrier Option

We shall look for solutions of the problem (2.10)-(2.12) in the following functional space:

$$u(x,\tau) \in C^0([0,\infty),\, H^{\frac{\nu}{2}+s}(0,a)), \quad |s| < 1/2.$$

This means that for each fixed $\tau \le 0$ $u(\cdot,\tau) \in H^{\frac{\nu}{2}+s}(0,a)$, and the function $F(\tau) := ||u(\cdot,\tau)||_{H^{\frac{\nu}{2}+s}}$ is continuous on $[0,\infty)$ with $\lim\limits_{\tau\to\infty} F(\tau) = 0$. Applying by Laplace transform (3.1) on the function $u(x,\tau)$ we have (at least for w with $\mathrm{Re} w > 0$) that

$$v(\cdot, w) \in H^{\frac{v}{2}+s}(0,a)$$

Further we have for the function

$$\widetilde{\Phi}_a^+(\xi, w) = (Fv)(-\xi, w)$$

the problem (5.11), (5.22), (5.23). This problem has a unique solution of the form (5.36) and this solution has $L_2(\mathbb{R}, \frac{\nu}{2}+s)$-norm bounded uniformly by w belonging to the region (5.10).

Thus applying the inverse Fourier Transform to the function $\Phi_a^+(-\xi, \dot{w})$ and then applying the inverse Laplace transform we obtain that the problem (2.10)-(2.12) has the solution of the following form

$$u(x,\tau) = \frac{1}{(2\pi)^2 i} \int_{L_{\theta_0}} \int_{-\infty}^{\infty} \widetilde{\Phi}_a^+(-\xi, w) e^{i\xi x + \tau w} d\xi dw \tag{1}$$

Here $\widetilde{\Phi}_a^+(-\xi, w)$ is given by (5.36) and the contour L_{θ_0} is the boundary of the sector $K_{\theta_0} := \{z \in \mathbb{C} | |\arg z| \le \frac{\pi}{2} + \theta_0\}$ for $\theta_0 > 0$ small enough.

Theorem 6.1. *Let $\nu \in (0,2)$, let the function $g(x) \in H^{-\frac{\nu}{2}+s}(0,a)$, for some $s \in (-1/2, 1/2)$ and let the be characteristic exponent under a EMM Q, the function $\psi^Q(\xi)$ (3.5), such that the symbol $c(\xi, w)$ (given by formula (5.7)) satisfies the conditions (5.2)-(5.5).*

Then the problem (2.10)-(2.12) has a unique solution in the space $C^0([0,\infty), H^{\frac{\nu}{2}+s)}(0,a))$ and this solution has the form (6.1).

This theorem follows from Theorem 5.1 and the fact that the function $e^{\tau w}$ decreases to cero as $e^{\tau \mathrm{Re} w}$. In fact, for w belonging to L_{θ_0}
Re $w < 0$ and Re $w \to -\infty$ if w passes along L_{θ_0}.

Now we are ready consider problem of finding the option price $U(x,t)$ (2.4). According to Theorem 2.13 of [15], $U(x,t)$ is a bounded solution of the problem (2.5)-(2.7).

Theorem 6.2. *Let $g(x) \in L_\infty(0,a)$ and let the process X_t satisfy the (ACP)-condition. Then the problem (2.10)-(2.12) (and problem (2.5)-(2.7) has no more than one solution.*

Suppose we have two bounded solutions $u_{1,2}(x,\tau)$ of the problem (2.10)-(2.12). Then the function $u_0(x,\tau) := u_2(x,\tau) - u_1(x,\tau)$ satisfies the following problem:

$$\begin{aligned} \frac{\partial u_0(x,\tau)}{\partial \tau} &+ (r - L_x^Q)u_0\ (x,\tau) = 0\ (x,\tau) \in (0,a)\times(0,\infty) && (2)\\ u_0(x,0) &= 0 \quad x \in (0,a) && (3)\\ u(x,\tau) &= 0 \quad x \in R\backslash(0,a), \tau \in (0,\infty) && (4)\end{aligned}$$

Applying to (6.2)-(6.4) the Laplace transform we obtain for the function

$$v_0(v,w) := (Lu_0)(x,w)$$

the following problem,

$$\begin{aligned} (-L_x^Q + r + w)v_0(x,w) = 0 \qquad & x \in (0,a) && (5)\\ v_0(x,w) = 0, \qquad & x \in R\backslash(0,a) && (6)\end{aligned}$$

The function $v_0(x,w)$ is bounded at least for all w with Re$w > 0$. The problem (6.5)-(6.6) is understood in sense of generalized functions:

$$(v(x), P_{[0,a]}(F^{-1}(\psi^Q(-\xi) + r + \overline{w})Fv_0) = 0 \tag{7}$$

where $v(x)$ is arbitrary function of $S(R)$ such that supp $v(x) \subset (0,a)$. Let $\{v_n\} \in S(R)$ be a sequence of functions with supp $v_n(x) \in (0,a)$ and such that $v_n(x) \to v_0(x,w)$ in the weak sense. Then we have from (6.7) that

$$(v_n(x), F^{-1}(\psi^Q(-\xi) + r + \overline{w})Fv_0) = 0$$

or equivalently

$$(Fv_n(x), (\psi^Q(-\xi) + r + \overline{w})Fv_0) = 0. \tag{8}$$

Introduce the sequence of numbers

$$\ell_n := (Fv_n, (\psi^Q(-\xi) + r + \overline{w})Fv_n)$$

According to (6.8),

$$\lim_{n\to\infty} \ell_n = 0.$$

But on the other hand according to Lemma 3.1

$$\begin{aligned} \mathrm{Re}\,\ell_\mathrm{n} &= \int_{-\infty}^{\infty} \mathrm{Re}(\psi^Q(-\xi) + r + \overline{w})|(Fv_n)(\xi)|^2 ds \geq \\ &\geq r\int_{-\infty}^{\infty} |(Fv_n)(\xi))^2 d\xi = r\int_{-\infty}^{\infty} |v_n(\xi)|^2 d\xi.\end{aligned}$$

That is, for n larger enough

$$\mathrm{Re}\,\ell_\mathrm{n} \geq \frac{r}{2}||v(\cdot,w)||^2_{L_2(\mathbb{R})};$$

That is $v_0(\xi, w) \equiv 0$ for all w with Re $w > 0$. Thus $u_0(x, \tau) \equiv 0$ and the theorem is proved.

We wish to obtain a bounded solution of the problem (2.5)-(2.7) or equivalently the problem (2.10)-(2.12).

For this we impose an aditional condition. Namely, let for some $s \in (-1/2, 1/2)$

$$\frac{\nu}{2} + s > \frac{1}{2} \tag{9}$$

It is well known that in this case

$$H^{\frac{\nu}{2}+s}(0,a) \subset C[0,a] \tag{10}$$

where $C[0,a]$ is the space of continuous functions on the segment $[0,a]$ and for the function $f(x) \in H^{\frac{\nu}{2}+s}(0,a)$ the following inequality holds,

$$\sup_{x\in[0,a]} |f(x)| \leq M||f||_{H^{\frac{\nu}{2}+s}} \tag{11}$$

with $M > 0$ constant.

Theorem 6.3. *Let all conditions of Theorem 6.1 and inequality (6.9) hold. Then the solution of the problem (2.10)-(2.12) is bounded.*

Proof. Acording Theorem 6.1 the problem (2.10)-(2.10) has a unique solution in the space $C^0([0,\infty), H^{\frac{\nu}{2}+s}(0,a))$ having the form (6.1). In virtue of (6.10) and (6.11) this solution is a bounded function in x uniformly in $t \in [0,\infty)$. $\square$

Finally suppose that $g(x)$ is a piecewise smooth function on the segment $[0,a]$. It is easy to see that in this case $g(x) \in H^{\mu}(0,a)$ for any $\mu < \frac{1}{2}$. For arbitrary $\frac{\nu}{2} \in (0,1)$ we always can choose $s \in [0, 1/2)$ such that condition (6.9) holds and moreover we have

$$\mu = -\frac{\nu}{2} + s < \frac{1}{2}.$$

Thus in this case according to Theorem 6.3 the problems (2.10)-(2.12) and (2.5)-(2.7) have bounded solutions. Since the Theorem 6.2 implies that this solution is unique, it has the form (6.1) and coincides with (2.4).

It should be noted that condition for the function $g(x)$ to be piecewise smooth holds very often in option theory.

7. Stability of the Solution Relatively Small Perturbation of Characteristic Function

Rewrite the equation (26) in the form

$$P^+(u_a(\xi)\Psi^+(\xi,w)) + P^+(\tilde{c}(\xi,w)\Psi_a^+(\xi,w)) = f^+(\xi). \tag{1}$$

Apply the projector P^+ to the equation (21)

$$P^+\overline{u}_a\Psi_a^+(\xi,w) = 0. \tag{2}$$

Rewrite (1)–(2) as a matrix equation

$$(T_{B_a}\vec{\Psi})(\xi) = \vec{F}^{+}(\xi) \tag{3}$$

where the vector functions

$$\vec{\Psi}(\xi) := \begin{pmatrix} \Psi^{+}(\xi) \\ \Psi_a^{+}(\xi) \end{pmatrix} \in L_2^{2+}(\mathbb{R}, s), \tag{4}$$

$$\vec{F}^{+}(\xi) := \begin{pmatrix} f^{+}(\xi) \\ 0 \end{pmatrix} \in L_2^{2+}(\mathbb{R}, s), \tag{5}$$

and the matrix Toeplitz operator is defined in the usual way,

$$T_{B_a} := \mathbb{P}^{+} B_a \Big|_{L_2^{2+}(\mathbb{R},s)} \tag{6}$$

with the matrix symbol

$$B_a(\xi) = \begin{pmatrix} u_a(\xi) & \dfrac{c(\xi,w)}{u_a(\xi)} \\ 0 & \end{pmatrix}. \tag{7}$$

Here the vector analytic projector

$$\mathbb{P}^{+} : L_2^{2}(\mathbb{R}, s) \to L_2^{2+}(\mathbb{R}, s)$$

is defined component-wise,

$$\mathbb{P}^{+} := \begin{pmatrix} P^{+} \\ P^{+} \end{pmatrix}.$$

It is obvious that the problems (26), (29), (30) and (3)–(5) are equivalent. Moreover the following result follows from Lemma 5.4.

It should be noted that the norm in space $L^2(\mathbb{R}, s)$ is define by usual way

$$||f_1, f_2||_{L_2^2(\mathbb{R},sw)} = (||f_1||^2_{L_2(\mathbb{R},s)} + ||f_2||^2_{L_2(\mathbb{R},s)})^{1/2}$$

Lemma 7.1. *The matrix Toeplitz operator T_{B_a} (7.6) is invertible in the space $L_2^{2+}(\mathbb{R}, s), |s| < 1/2$, if and only if the operator D_{u_a} (31) is invertible in the space $L_2^{+}(\mathbb{R}, s)$.*

Thus the invertibility of the operator T_{B_a} follows from Theorem 5.1, i).

Theorem 7.1. *Let the function $c(\xi, w)$ (7) satisfy conditions (3)–(6). Then the operator T_{B_a} is invertible and*

$$\|T_{B_a}^{-1}\|_{L_2^2(\mathbb{R})} \le M\varepsilon^{-1}$$

where ε is given by (8), (9), and $M > 0$ is constant.

Thus we can write the solution of the option problem with the help of the operator $T_{B_a}^{-1}$. Indeed, under the hypotheses of Theorem 7.1 the solution of equation (3) has the form

$$\begin{pmatrix} \Psi^{+}(\xi) \\ \Psi_a^{+}(\xi) \end{pmatrix} = T_{B_a}^{-1} \begin{pmatrix} f^{+} \\ 0 \end{pmatrix}.$$

So the formula (6.1) can be rewritten in the form

$$\mathcal{U}(x,t) = \frac{1}{(2\pi)^2 i} \int\limits_{R_\sigma} \int\limits_{-\infty}^{\infty} T_{B_a}^{-1} \left(\begin{array}{l} f^+(\xi) \\ 0 \end{array} \right)\Bigg|_2 e^{(T-t)w - i\xi x} d\xi dw \tag{8}$$

where $\vec{F}(\xi)\Big|_2$ denotes the second component of the vector function $\vec{F}(\xi)$. Thus practical (approximate) solution of the equation (3) is an important problem. The following reasoning can be considered as a basis for some algorithms of approximate solution.

With equation (3) consider

$$T_{B_a^*} \vec{\Psi} = \vec{F}_0^* \tag{9}$$

where the approximate symbol of the Toeplitz operator $B_a^*(\xi)$ and right-hand member $\vec{F}_0^*$ have the forms

$$B_a^*(\xi) = \left(\begin{array}{ll} u_a^*(\xi) & \dfrac{c^*(\xi,w)}{u_a^*(\xi)} \\ 0 & \end{array} \right); \quad \vec{F}_0^*(\xi) = \left(\begin{array}{l} f^*(\xi); \\ 0 \end{array} \right)$$

with the components satisfying the following conditions

$$\sup_{\xi \in \mathbb{R}} |u_a(\xi) - u_a^*(\xi)| \le \delta_0, \tag{10}$$

$$\sup_{\xi \in \mathbb{R}} |\widetilde{c}(\xi,w) - c^*(\xi,w)| \le \delta_0, \tag{11}$$

and

$$\|f^+ - f^{*+}\|_{L_2(\mathbb{R},s)} \le \delta_1 \tag{12}$$

where the numbers $\delta_0, \delta_1 > 0$ are sufficiently small.

The following theorem is a standard fact from the theory of Toeplitz operators ([40]–[41]).

Theorem 7.2. *Let the function $c(\xi,w)$ (7) satisfy conditions (1)–(6). Then for δ_0 small enough the operator $T_{B_a^*}$ is invertible, equation (9) has an unique solution $\vec{\Psi}^*(\xi)$ and the following estimate holds,*

$$\|\vec{\Psi} - \vec{\Psi}^*\|_{L_2^2(\mathbb{R},s)} \le M_0\delta + M_1\delta_1. \tag{13}$$

In particular,

$$\|\Psi_a^+ - \Psi_a^{*+}\|_{L_2(\mathbb{R},s)} \le M_0\delta + M_1\delta_1$$

where $\vec{\Psi} = \left(\begin{array}{l} \Psi^+ \\ \Psi_a^+ \end{array} \right)$ *and* $\vec{\Psi}^* = \left(\begin{array}{l} \Psi^{*+} \\ \Psi_a^{*+} \end{array} \right)$ *are the solutions of equations (3) and (7.9) respectively, and $M_0, M_1 > 0$ are independent of δ_0, δ_1.*

Proof. According to Theorem 7.1, the operator T_{B_a} is invertible. Therefore if δ_0 is small enough, then the operator $T_{B_a^*}$ is invertible also and

$$\|T_{B_a}^{-1} - T_{B_a^*}^{-1}\|_{L_2^2(\mathbb{R},s)} \le C\delta_0$$

where $C > 0$ is independent of δ_0. Thus equation (9) has the unique solution $\vec{\Psi}^* = T_{B_a^*}\vec{F}^*$ and we have the following inequalities,

$$\begin{aligned}\|\vec{\Psi} - \vec{\Psi}^*\|_{L_2^2(\mathbb{R},s)} &= \|T_{B_a}^{-1}\vec{F} - T_{B_a^*}^{-1}\vec{F}^*\|_{L_2^2(\mathbb{R},s)} \\ &= \|(T_{B_a}^{-1} - T_{B_a^*}^{-1})\vec{F} + T_{B_a^*}^{-1}(\vec{F} - \vec{F}^*)\|_{L_2^2(\mathbb{R},s)} \\ &\leq (C\|\vec{F}\|_{L_2^2(\mathbb{R},s)})\delta_0 + (\|T_{B_a^*}^{-1}\|_{L_2^2(\mathbb{R},s)})\delta_1.\end{aligned}$$

Denote $M_0 := C\|\vec{F}\|_{L_2^2(\mathbb{R},s)}$ and $M_1 = 2\|T_{B_a}^{-1}\|_{L_2^2(\mathbb{R},s)}$. Then for δ_0, δ_1 small enough we have the evaluation (13). ■

Thus the approximate solution of our option problem can be written in the form (see (8))

$$\mathcal{U}^*(x,t) = \frac{1}{(2\pi)^2 i} \int\limits_{R_\sigma} \int\limits_{-\infty}^{\infty} T_{B_a^*}^{-1} \left(\begin{array}{c} f^{*+}(\xi) \\ 0 \end{array} \right) \Bigg|_2 e^{(T-t)w - i\xi x} d\xi dw. \tag{14}$$

This formula can serve as the basis for an algorithm for the approximate solution of the double barrier option problem. We will present this algorithm in future work.

8. Conclusion

In this article we treat some power cases of characteristic functions (see (12)–(13)). These cases involve wide classes of Lévy processes which are used in option theory. However, there exist many other cases which could be considered with the help of the methods worked out in this article.

1. The case $\sigma > 0$ is important because it corresponds to the processes with non trivial Gaussian components. This case can be realized as the case $\nu < 2$ considered in these notes.
2. The case $\sigma = 0$, $\mu \neq 0$ and $0 < \nu < 1$ (see (14), (15), (16), (20), (21)).
3. Logarithmic cases (16) and (22) if $\mu = 0$.
4. Power logarithmic case (17).
5. Rational case (27). In this case not only the solvability theory can worked out but one can obtain the solution in explicit form.
6. Periodic case. The Poisson process generates a periodic characteristic function (24). It is interesting to get explicit formulae and to analyze them in this case. (25) is very interesting also because here X_t is sum of a Gaussian process and a discrete-jumping process. In this area the theory of matrix Toeplitz operators with periodic and almost periodic symbols (worked out by Karlovich-Spitkovsky-Böttcher see [45]) could be applied.
7. General case. According to a famous result ([28, p.13]) for an arbitrary triplet (a, γ, Π) with measure Π satisfying (3) there exists a Lévy process X_t with this characteristic triplet. The condition (3) is quite general . Thus there exist Lévy processes with characteristic function having discontinuities of the first

type at infinity, semi almost periodic discontinuities and so on. It is very interesting to consider the double barrier option problem for the general case when characteristic function has the form (2)–(3).

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References

[1] Kunimoto N., Ikeda M. Pricing options with curved boundaries. Math. Finance 2, 275–298, 1992.

[2] Geman H. and Yor M. Pricing and Hedging double-barrier options: A probabilistic approach. Math. Finance **6:4**, 365–378, 1996.

[3] Sidenius J. Double Barrier Options - Valuating by Path Counting. J. Computat. Finance, 1998, V1, No. 3.

[4] Pelsser A. Pricing double barrier options using Laplace transform. Finance Stochastics **4:1**, 95–104, 2000.

[5] Baldi P, Caramellino L, Iovino M.G. Pricing general Barrier Options using sharp larger deviations. Math. Finance 9:4, 291-322, 1999.

[6] Hui C.H. One-touch double barrier binary option values. Appl. Financial Econ. **6**, 343–346, 1996.

[7] Hui C.H. Time dependent barrier option values. J. Futures Markets, 17, 667–688, 1997.

[8] Wilmott P., Dewynne J., Howison S. Option pricing: Mathematical models and computations. Oxford: Oxford Financial Press, 1993.

[9] Raible R. Lévy processes in Finance: Theory, Numerics, and Empirical Facts. Dissertation. Mathematische Fakultät, Universität Freiburg im Breisgau, 2000.

[10] Barndorff-Nielsen O.E. Processes of normal inverse Gaussian Type. Finance and Stochastics **2**, 41–68, 1998.

[11] Madan D.B., Carr P., and Chang E.C. The variance Gamma process and option pricing. European Finance Review **2**, 79–105, 1998.

[12] Eberlein E. Application of generalized hyperbolic Lévy motions to Finance. In: Lévy processes: Theory and applications, O.E. Barndorff-Nielsen, T. Mikosh and S. Resnik (Eds.), Birkhäuser, 319–337, 2001.

[13] Bouchaud J.-P. and Potters M. Theory of financial risk. Cambridge University Press, Cambridge.

[14] Matacz A. Financial modelling and option theory with the truncated Lévy process. Intern. Journ. Theor. and Appl. Finance **3:1**, 143–160, 2000.

[15] Boyarchenko S.I., Levendorskiĭ Sergei. Non-Gaussian Merton-Black-Scholes theory. Advanced Series on Statistical Science and Applied Probability **9**, World-Scientific, Singapore, 2002.

[16] Boyarchenko S.I., Levendorskiĭ S.Z. On rational pricing of derivative securities for a family of non-Gaussian processes. Preprint 98/7, Institut für Mathematik, Universität Potsdam, Potsdam.

[17] Boyarchenko S.I., Levendorskiĭ S.Z. Option pricing and hedging under regular Lévy processes of exponential type. In: Trends in Mathematics. Mathematical Finance, M. Kohlman and S. Tang (Eds.), 121–130, 2001.

[18] Levendorskiĭ S.Z. and Zherder V.M. Fast option pricing under regular Lévy processes of exponential type. Submitted to Journal of Computational Finance, 2001.

[19] Boyarchenko S.I., Levendorskiĭ S.Z. Option pricing for truncated Lévy processes. Intern. Journ. Theor. and Appl. Finance **3:3**, 549–552, 2000.

[20] Boyarchenko S.I., Levendorskiĭ S.Z. Perpetual American options under Lévy processes. SIAM Journ. of Control and Optimization, Vol. 40 (2002) no. 6, pp. 1663–1696

[21] Mordecki E. Optimal stopping and perpetual options for Lévy processes. Talk presented at the 1 World Congress of the Bachelier Finance Society, June 2000.

[22] Boyarchenko S.I., and Levendorskiĭ S.Z. Barrier options and touch-and-out options under regular Lévy processes of exponential type. Annals of Applied Probability, Vol. 12 (2002), no. 4, pp. 1261–1298.

[23] Cont R., Voltchkova E. Integro-differential equations for Options prices in exponential Lévy models. Finance Stochastics. 9, 299-325 (2005).

[24] Grudsky S.M. Convolution equations on a finite interval with a small parameter multiplying the growing part of the symbol. Soviet Math. (Iz. Vuz) **34**, 7, 7–18, 1990.

[25] Böttcher A., Grudsky S.M. On the condition numbers of large semi-definite Toeplitz matrices. Linear Algebra and its Applications **279**, 1–3, 285–301, 1998.

[26] Grudsky S.M., Mikhalkovich S.S. Semisectoriality and condition numbers of convolution operators on the large finite intervals. Integro-differential operators, Proceedings of different universities, Rostov-on-Don, 5, 78–87, 2001.

[27] Grudsky S.M., Mikhalkovich S.S., and E. Ramíirez de Arellano. The Wiener-Hopf integral equation on a finite interval: asymptotic solution for large intervals with an application to acoustics. Proceedings of International Workshop on Linear Algebra, Numerical Functional Analysis and Wavelet Analysis, India, 2002 (submitted).

[28] Bertoin J. Lévy processes. Cambridge University Press, Cambridge, 1996.

[29] Shiryaev A.N. Essentials of stochastic Finance. Facts, models, theory. World Scientific, Singapore Jersey London Hong Kong, 1999.

[30] Karatzas I. and Shreve S.E. Methods of mathematical Finance. Springer-Verlag, Berlin Heidelberg New York, 1998.

[31] Delbaen F. and Schachermayer W. A general version of the fundamental theorem of asset pricing. Math. Ann. **300**, 463–520, 1994.

[32] Eberlein E. and Jacod J. On the range of options prices. Finance and Stochastics **1**, 131–140, 1997.

[33] Fölemer H. and Schweizer M. Heedging of contingent claims under incomplete information. In: Applied Stochastic Analysis, M.H.A. Davies and R.J. Elliot (eds.), New York, Gordon and Bleach, 389–414, 1991.

[34] Keller U. Realistic modelling of financial derivatives. Dissertation. Mathematische Fakultät, Universität Freiburg im Breisgau, 1997.

[35] Kallsen J. Optimal portfolios for exponential Lévy processes. Mathematical Methods of Operations Research **51:3**, 357–374, 2000.

[36] Madan D.B. and Milne F. Option prising with VG martingale components. Mathem. Finance **1**, 39–55, 1991.

[37] Eberlein E, Keller U. and Prause K. New insights into smile, mispricing and value at risk: The hyperbolic model. Journ. of Business **71**, 371–406, 1998.

[38] Eskin G.I. Boundary problems for elliptic pseudo-differential equations. Nauka, Moscow, 1973 (Transl. of Mathematical Monographs, 52, Providence, Rhode Island: Amer. Math. Soc., 1980).

[39] Noble B. Methods based on the Wiener-Hopf technique for the solution of partial differential equations. International Series of Monographs on Pure and Applied Mathematics **7**, Pergamon Press, New York-London-Paris-Los Angeles, 1958.

[40] Gohberg I. and Krupnik N.Ya. Introduction to the Theory of One-dimentional Singular integral Operators. "Shtiintsa", Kishinev, 1973 (Transl. One-dimentional Linear Singular Integral Equations. Vol. I. Introduction, Vol II General Theory and Applications, Translated from the 1979 German Translation. Operator theory: Advances and Applications, 53 and 54. Birkhäuser Verlag, Basel, 1992).

[41] Böttcher A. and Silbermann B. Analysis of Toeplitz Operators. Springer-Verlag, Berlin, 1990.

[42] Sarason D. Toeplitz operators with semi-almost-periodic symbols. Duke Math. J. **44**, 2, 357–364, 1977.

[43] Akhiezer N.I. Lectures on Approximation Theory. Second, revised and enlarged edition, "Nauka", Moscow, 1965 (Transl. of first edition: Theory of Approximation, Frederick Ungar Publishing Co., New York, 1956).

[44] Dybin V.B., Grudsky S.M. Introduction to the theory of Toeplitz operators with infinite index. Birkhäuser Verlag, Basel-Boston-Berlin, Operator Theory: Advances and Applications, 2002.

[45] Böttcher A, Karlovich Yu. I Spitkovsky I.M. Convolution Operators and Factorization of Almost Periodic Matrix Functions. Operator Theory: Advances and Application. Vol. 131, Birkhäuser Verlag, 2002.

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